MATH UN1207, Honors Math A

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Contents

1 September 4, 2019

1.1 Statements

Definition. *A statement (or proposition) is an assertion that its true or false (but not both).*

Ex
P = "Ringo Starr is alive"
Q = "The earth is flat"

Definition. *The truth value of a statmeent is T (if true) and F (if false)*

Definition. *The negation of a statement P is the statement "P is false" (∼P)*

Ex
∼P = "Ringo Starr is dead"
∼Q = "The earth is not flat"

Remark. $\sim(\sim P) = P$

Definition. *The conjunction of two statements P, Q is the statement "P and Q" ($P \wedge Q$)*
i.e. The truth value of $P \wedge Q$ is T if P is T and Q is T, and F otherwise.

Proposition. $P \wedge(\sim P) = F$

P	∼P	P ∧(∼P)
T	F	F
F	T	F

Definition (Disjunction)**.** *P or Q ($P \vee Q$) is F if P is F and Q is F, and T otherwise. "or" in math is always inclusive.*

Proposition. $(P \wedge Q) \vee (\sim P \vee \sim Q) = T$

P	Q	P ∧ Q	∼P ∨ ∼ Q	(P ∧ Q) ∨ (∼P ∨ ∼ Q)
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

1.2 Conditionals

Definition (Conditional)**.** *P implies Q or if P then Q, or $P \Rightarrow Q$, is $Q \vee (\sim P)$. i.e. if P is T, then Q is T.*

Remark. *If P is F, then we say $P \Rightarrow Q$ is "vacuously true"*

Proposition. *$(P \wedge Q) \Rightarrow P$, $(P \wedge Q) \Rightarrow Q$, $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$*

Definition (Biconditional)**.** *P iff Q or P if and only if Q or $P \iff Q$ is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$*

P	Q	P $\iff$ Q
T	T	T
T	F	F
F	T	F
F	F	T

Proposition. *1 - Suppose P and $P \Rightarrow Q$. Then Q.*
2 - Suppose $P \Rightarrow Q$ and $\sim Q$. Then $\sim P$.
3 - Suppose $P \Rightarrow Q$ and $P \Rightarrow \sim Q$. Then $\sim P$.
4 - Suppose $P \vee Q, P \Rightarrow R$, and $Q \Rightarrow R$. Then R.

Proof of 3. (P $\Rightarrow$ Q) $\wedge$ (P $\Rightarrow\sim$ Q) $\Rightarrow \sim$ P

P	Q	P $\Rightarrow$ Q	P $\Rightarrow\sim$ Q	(P $\Rightarrow$ Q) $\wedge$ (P $\Rightarrow\sim$Q)	Whole
T	T	T	F	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	T	T	T

□

$$\text{De Morgan's Laws} \begin{cases} \sim (P \wedge Q) \iff \sim P \vee \sim Q \\ \sim (P \vee Q) \iff \sim P \wedge \sim Q \end{cases}$$

Definition (Contrapositive)**.** *$(P \Rightarrow Q) \iff (\sim Q \Rightarrow \sim P)$*

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2.1 Predicates

Definition. *A predicate $P(x)$ is a family of statements depending on a variable x*

Ex
$P(x) =$ "x is a banana"
$Q(x) =$ "$x > 7$ "

Existential Quantifier:
$\exists x \mid P(x)$: "there exists an x such that $P(x)$"
Universal Quantifier:
$\forall x\, P(x)$: "for all x, $P(x)$"

e.g. $\forall x, Q(x)$ is false ($Q(x)$ as above)
$\forall x(P(x) \vee \sim P(x)))$ is always true

e.g.

$$\sim (\forall x P(x)) \iff \exists x(\sim P(x))$$

$$\sim (\exists P(x)) \iff \forall x(\sim P(x))$$

Remark. *To prove $\exists x P(x)$, just find an x such that P(x). To prove $\forall x P(x)$, write something like "take an x..."*

2.2 Sets

A set is a collection of objects.
e.g.

$$S = \{1, 2, 4\}, T = \{\{1\}, 2, \text{water}\}$$

$\{1\} \in T$, but $1 \notin T$. "1 is not an element of T"

$\mathbb{N}$ = set of natural numbers $\{1, 2, 3, ...\}$
$\mathbb{Z}$ = set of integers $\{..., -2, -1, 0, 1, 2, ...\}$
$\mathbb{Q}$ = rational numbers
$\mathbb{R}$ = real numbers
$\mathbb{C}$ = complex numbers

Definition. *$\forall x \in S, P(x)$ just means $\forall x(x \in S \Rightarrow P(x))$*

Definition (Set Inclusion)**.** *$S \subseteq T$ (S is a subset of T) if $\forall x \in S, x \in T$*

Remark. *In definitions, we write "if" instead of "if and only if" even though the latter is what we mean.*

Definition. *$S = T$ if $S \subset T$ and $T \subset S$*

Warning - Order matters with quantifiers!
$\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \mid m + n = 0$ is TRUE
$\exists m \in \mathbb{Z} \mid \forall n \in \mathbb{Z}, m + n = 0$ is FALSE

Proposition. *IF $S \subset T$ and $T \subset U$, then $S \subset U$*

Proof. We know $\forall x \in S, x \in T$ and $\forall yinT, y \in U$. Therefore, $\forall x \in S, x \in U$. Thus, $S \subset U$. ☐

2.3 Axioms for Sets

1. There exists a set
2. "Axiom of Specification" (how to take subsets)
   Given set S and any predicate $P(x)$, there exists a set T such that
   (a) $T \subset S$
   (b) $\forall x, (x \in T \iff P(x))$

   We write $T = \{x \in S \mid P(x)\}$.

   Take $Q(x)$ = "$x \neq x$"(always false). Then, take S any set $= \{x \in S \mid Q(x)\} = \emptyset$ (the empty set).

2.4 Russell's Paradox

If we take axiom of specification w/o picking S, we get a contradiction. Take $P(x)$ = "$x = x$"(always true). Stengthened axiom $\Rightarrow$ get a set of all sets $\mathcal{V}$. We will show a contradiction.
Let $T = \{S \in \mathcal{V} \mid S \notin S\}$. Is $T \in T$?. If $T \in T \Rightarrow T$ really bad $\Rightarrow T \notin T \Rightarrow\Leftarrow$. If $T \notin T \Rightarrow T$ is not really bad $\Rightarrow T \in T \Rightarrow\Leftarrow$. We conclude that there is no "set of all sets".

2.5 Axioms Cont.

3. Axiom of Unions
   Given sets S, T, there exists $S \cup T$ such that $\forall x, x \in S \cup T \iff x \in S$ or $x \in T$.

3'. Axiom of Intersection
   Given sets S, T, there eixsts $S \cap T$ such that $\forall x (x \in S \cap T) \iff x \in S$ and $x \in T$

Theorem 2.1. $A \cup B = B \cup A$

Theorem 2.2. $(A \cup b) \cup C = A \cup (B \cup C)$

Theorem 2.3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof of 2.3. Need to show: $\forall x, x \in LHS \iff x \in RHS$

$$\begin{aligned} x \in LHS &\iff x \in A \cap (B \cup C) \\ &\iff x \in A \wedge x \in (B \cup C) \\ &\iff x \in A \wedge (x \in B \vee x \in C) \\ &\iff (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\ &\iff x \in A \cap B \vee x \in A \cap C \\ &\iff x \in (A \cap B) \cup (A \cap C) \\ &\iff x \in RHS \end{aligned}$$

□

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3.1 More De Morgan

Definition. *Let S, A be sets. Then the complement S-A or* $S\setminus A$ *is* $\{x \in S \mid x \notin A\}$

There are two more De Morgan laws.

1. $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$
2. $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$

3.2 Power Sets

4. For all sets A, there exists a set $\mathcal{P}(A)$, the power set of A, such that its elements are precisely the subsets of A.

$$\forall B, B \subseteq A \Leftrightarrow B \in \mathcal{P}(A)$$

Ex $A = \{1, 2\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Proposition. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

Proof. Similar to other set equality proofs.

$$\begin{aligned} S \in \mathcal{P}(A \cap B) &\iff S \subseteq (A \cap B) \\ &\iff \forall x \in S, (x \in A \cap B) \\ &\iff \forall x \in S, (x \in A \text{ and } x \in B) \\ &\iff \forall x \in S, x \in A \text{ and } \forall x \in S, s \in B \\ &\iff S \subseteq A \text{ and } S \subseteq B \\ &\iff S \in \mathcal{P}(A) \text{ and } S \in \mathcal{P}(B) \\ LHS &\iff S \in \mathcal{P}(A) \cap \mathcal{P}(B) \end{aligned}$$

□

3.3 Cartesian Products

"Def": An ordered pair is a list (a, b) where a, b are math objects. We say $(a, b) = (a', b')$ if $a = a'$ and $b = b'$.
Ex $(1, 2) \neq (2, 1)$

5. Axiom of Products: Givens sets A, B, there exists a set $A \times B$ whose elements are exactly the pais (a, b) with $a \in A, b \in B$.

$$\forall x, x \in A \times B \iff x = (a, b) \text{ with } a \in A, b \in B$$

Remark. *This axiom is actually not necessary*

Proposition. $A \times (B \cup C) = A \times B \cup A \times C$

Proof. Same format as any other set equality proof.

$$\begin{aligned}
(a, b) \in LHS &\iff a \in A \text{ and } b \in (B \cup C) \\
&\implies a \in A \text{ and } (b \in B \text{ or } b \in C) \\
&\iff (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C) \\
&\iff (a, b) \in A \times B \text{ or } (a, b) \in A \times C \\
&\iff (a, b) \in (A \times B) \cup (A \times C)
\end{aligned}$$

□

3.4 Functions

"Def" Let S, T be sets. A function $F : S \to T$ (from S to T) is a rule assigning an element of T to each element of S. i.e. If $s \in S$, this element of T is denoted $f(s)$.

Remark. *We can make a non-fake def of a function $S \to T$ by defining it as a subset of $S \times T$.*

Remark. *We write $f : S \to T$, "f is a function from S to T", "f maps S to T", and $x \mapsto x^2$, "x maps to x^2".*

Definition. *If $f : S \to T$, then S is the domain and T is the codomain.*

Definition. *The graph of $f : S \to T$, (sometimes denoted $\Gamma(f)$) is $\{(x, y) \in S \times T \mid y = f(x)\}$.*

Definition. *Two functions f: $S \to T$ and $g : S' \to T'$ are equal if $S = S'$, $T =$ T', and $\forall s \in S, f(s) = g(s)$.*

Definition. *If $f : S \to T$ and $g : T \to U$ are functions, their composition is the function $g \circ f : S \to U$ such that $(g \circ f)(s) = g(f(s)) \forall s \in S$.*

Definition. *If S is set, then id_s: $S \to S$ is called the identity function*

3.5 Injectivity and Surjectivity

Definition. *$f : S \to T$ is injective if wheneber $f(s) = f(s'), s = s'$.*

Definition. *$f : S \to T$ is surjective if $\forall t \in T, \exists s \in S$ such that $f(s) = t$.*

Proposition. *id_s is injective and surjective/*

Proof. If $\mathrm{id_s(s)} = \mathrm{id_s}(s'), s = s'$. This proves injectivity. $\forall s \in S$, we have $\mathrm{d_s(s)}$ = s. This proves surjectivity. Thus, we have both. □

Remark. *$f : S \to T$ injective $\iff \forall t \in T$, there is at* <u>*most*</u> *one preimage(at most one $s \in S$ such that $f(s) = t$). f surj $\iff \forall t \in T$, there is at least one element of preimage.*

Definition. *$f : S \to T$ is bijective if it is injective and surjecitve.*

<u>Ex/Prop</u>

1. $f : S \to T, g : T \to U$, both inj.
   (a) $\Rightarrow g \circ f$ inj.
   (b) $\Rightarrow f$ inj.
2. $f : S \to T, g : T \to U$ surj.
   (a) $g \circ f$ surj.
   (b) g surj.

3.6 Inverses

Definition. *If $f : S \to T$ is a function, an* <u>*inverse*</u> *to f is a function $g : T \to S$ such that*

1. $g \circ f = \mathrm{id_S}$: $S \to S$
2. $f \circ g = \mathrm{id_T}$: $T \to T$

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4.1 Inverses Cont.

Theorem 4.1. *$f : S \to T$ has an inverse if and only if it is bijective.*

Proof. ($\Rightarrow$) by homework problem
($\Leftarrow$) We know by definition of bijectivity, $\forall t \in T, \exists s \in S \mid f(s) = t$ and $\forall s, s' \in S, f(s) = f(s') \Rightarrow s = s'$. Define $g : T \to S$.

Lemma 4.2. *$f(s) = t \iff g(t) = s$*

Lemma can be proven by definition of g. Lemma $\Rightarrow$ if $g(t) = s'$, then $f(s') = t$. Thus, $t = f(s) = f(s')$. By injectivity of $f, s = s'$.

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) \\ &= f(s) \\ &= t\end{aligned}$$

$$
\begin{aligned}
(g \circ f)(s) &= g(f(s)) \\
&= g(t) \\
&= s
\end{aligned}
$$

□

Remark. *By definition, if g is inverse to f, then f is inverse to g.*

Proposition. *If g, g' are inverses to f, then $g = g'$. (Inverses are unique)*

Proof. Take any $t \in T$.

$$
\begin{aligned}
g'(t) &= (g' \circ id_t)(t) \\
&= (g' \circ (f \circ g))(t) \\
&= ((g' \circ f) \circ g)(t) \\
&= (id_s \circ g)(t) \\
&= g(t)
\end{aligned}
$$

□

Definition. *If $f : S \to T$ and $U \subseteq S$, then the image of U (under f), denoted $f(U)$, is $\{t \in T \mid \exists s \in U \text{ with } f(s) = t\} = \{f(s) \mid \overline{s \in U}\}$.*

Definition. *If $f : S \to T$ and $V \subseteq T$, the preimage of V under f, denoted $f^{-1}(V$, is $\{s \in S \mid f(s) \in V\}$.*

4.2 Numbers

We will axiomatize $\mathbb{R}$.

Definition. *A binary operation on a set S is a function $S \times S \to S$.*

Definition. *A relation on S is a subset of $S \times S$.*

Assumption: There exists a set $\mathbb{R}$, equippied with two binary relations $+$, $\bullet$, one relation $>$, and two elements 0, 1 satisfying the following axioms.

4.3 Axioms

All of the axioms are $\forall x, y, z \in \mathbb{R}$, unless otherwise noted.

1. Commutativity
   $x + y = y + x$ $\qquad$ $x \cdot y = y \cdot x$
2. Associativity
   $x + (y + z) = (x + y) + z$ $\qquad$ $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
3. Distributivity
   $x \cdot (y + z) = x \cdot y + x \cdot z$

4. Identity Elements
   $0 \neq 1 \qquad 0 + x = x \qquad 1 \cdot x = x$

5. Additive Inverse
   $\forall x \in \mathbb{R}, \exists w \in \mathbb{R} \mid w + x = 0$. We can denote w by $-x$.

Note: Axioms 1-5 are a commutative ring.

6. Multiplicative Inverse
   $\forall x \neq 0 \in \mathbb{R}, \exists w \in \mathbb{R} \mid w \cdot x = 1$. We can denote w by $\frac{1}{x}, x^{-1}$, etc.

Note: Axioms 1-6 are a field.

e.g. $\frac{\mathbb{Z}}{n\mathbb{Z}} = \{0, 1, ..., n-1\}$ with modular arithmetic is a commutative ring and is a field iff n is prime.
Notation: $x + (-y) =: x - y$
$x \cdot \frac{1}{y} = \frac{x}{y}$

7. Order Axiom 1
   If $x > 0$ and $y > 0$, then $x + y > 0$ and $x \cdot y > 0$. ("add + mult preserve the order").

8. Order Axiom 2
   If $x \neq 0$, then $x < 0$ or $x > 0$ but not both. ("trichotomy")

9. Order Axiom 3
   not $0 > 0$ ($0 \not> 0$)

10. Order Axiom 4
    If $x > y$, then $x + z > y + z$.

Note: Axioms 1 - 10 are an ordered field. (e.g. $\mathbb{Q}, \mathbb{R}$)

Proposition. *High School Algebra*

Proof. exercise □

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5.1 Application of Axioms

Proposition (Multiplicative Cancellation)**.** *Given an ordered field* $\mathbb{R}, \forall a, b, x \in \mathbb{R}$

$$xa = xb, x \neq 0 \Longrightarrow a = b$$

Proof. By axiom of multiplicative inverse, $\exists w \in \mathbb{R}$ with $wx = 1$. Since $xa = xb$, we can multiply both sides by w to obtain $wxa = wxb$. This statement is equal to $(wx)a = (wx)b$ by associativity. Then, $1a = 1b$. Thus, $a = b$ by the axiom of identity. □

Theorem 5.1 (Trichotomy). *$\forall a, b \in \mathbb{R}$, exactly one of the following is true:*

1. $a > b$
2. $b > a$
3. $a = b$

Lemma 5.2. $\forall a, b \in \mathbb{R}, a > b \iff a - b > 0$

Proof. Do each direction separately.
($\Rightarrow$)

$$\begin{aligned} a > b &\Rightarrow a + (-b) > b + (-b) \text{ by axiom of add. inverse and axiom 10} \\ &= a - b > 0 \text{ by def of "-"} \checkmark \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} a - b > 0 &\Rightarrow (a - b) + b > 0 + b \text{ by axiom 10} \\ &\Rightarrow a + (-b + b) > b \text{ by associativity and add. identity} \\ &\Rightarrow a + (b + -b) > b \text{ by commutativity} \\ &\Rightarrow a + 0 > b \text{ by add. inverse} \\ &\Rightarrow 0 + a > b \text{ by commutativity} \\ &\Rightarrow a > b \text{ by add. identity} \end{aligned}$$

□

Lemma 5.3. $\forall a, b \in \mathbb{R}, b > a \iff 0 > a - b$

Proof. Either copy Lemma 5.2 (almost) or $b > a \iff b - a > 0$ by Lemma 5.2 then show $b - a > 0 \iff 0 > a - b$. □

Lemma 5.4. $\forall a, b \in \mathbb{R}, a = b \iff a - b = 0$

Proof. Simple

$$\begin{aligned} a = b &\Rightarrow a + (-b) = b + (-b) \\ &\Rightarrow a - b = 0 \\ &\Rightarrow (a - b) + b = 0 + b \\ &\Rightarrow (a) + (-b + b) = b \\ &\Rightarrow a + (b + -b) = b \\ &\Rightarrow a + 0 = b \\ &\Rightarrow a = b \end{aligned}$$

□

Proof of 5.1. Let $x = a - b$. Either $x = 0$ or $x \neq 0$ but not both. By Axiom 8, if $x \neq 0$, either $x > 0$ or $0 > x$ but not both. By Axiom 9, if $x = 0$ then NOT $x > 0, 0 > x$. This implies that exactly one of $x = 0, x > 0, 0 > x$ is true. Lemmas 5.2, 5.3, 5.4 tell us that $x = 0 \iff 5.4, x > 0 \iff 5.2, 0 > x \iff 5.3$. ☐

Exercises to try on own

- $\forall x, 0 \cdot x = 0$
- $1 > 0$ (tricky)

5.2 Natural Numbers

Definition. *$S \subseteq \mathbb{R}$ is an inductive set if*

1. $0 \in S$
2. $\forall x \in S, x + 1 \in S$

Ex $\{x \in \mathbb{R} \mid x \geq 0\}$ is inductive (by above exercise)

Definition. *A natural number is an $x \in \mathbb{R}$ such that x is a number of every inductive set.*

The set of natural numbers is called $\mathbb{Z}_{\geq 0}$.

Theorem 5.5 (Mathematical Induction)**. ??** *Let $P(n)$ be a predicate defined on $\mathbb{Z}_{\geq 0}$ such that*

1. *$P(0)$ is true (base case)*
2. *$\forall n \in \mathbb{Z}, P(n) \Rightarrow P(n+1)$*

Then, $\forall n \in \mathbb{Z}_{\geq 0}, P(n)$ is true.

Proof. $\{n \in \mathbb{R} \mid P(n)\}$ is an inductive set. Hence $\mathbb{Z}_{\geq 0} \subseteq S$ by definition of the natural numbers. ☐

Proposition. $\forall n \in \mathbb{Z}_{\geq 0}, n^2 > n$

Proof. Base Case $n = 0 : 0 \geq 0$✓. Inductive step: Assume $n^2 \geq n$.

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 2n + 1 \\ &\geq n + n + 1 \\ &\geq n + 1\end{aligned}$$

☐

Proposition. $\forall n \in \mathbb{Z}_{\geq 0}, 0 + 1 + 2 + ... + n = \frac{n(n+1)}{2}$

Proof. Base Case $n = 0 : 0 = 0\checkmark$. Inductive step: Assume $0 + ... + n = \frac{n(n+1)}{2}$. Add $n + 1$ to both sides.

$$\begin{aligned} 0 + ... + n + n + 1 &= \frac{n(n+1)}{2} + n + 1 \\ &= n + 1\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

□

Definition. *A positive integer is a natural number that is not 0.*

Definition. *An integer is the difference of two natural numbers (called $\mathbb{Z}$).*

Definition. *A rational number is a quotient of two integers($\mathbb{Q}$).*

<u>"Principle"</u> of recursive definition: For any set S, a function $f : \mathbb{Z}_{\geq 0} \to S$ may be specified by a choice of $f(0)$ and a function expressing $f(n)$ in terms of $f(m), m < n$.

<u>"Proof"</u>
$\{n \in \mathbb{Z}_{\geq 0} \mid \forall m < n$, the above data determine $f(n)$ uniquely$\}$ is an inductive set.
<u>Ex</u>
Define $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. $f(0) = 0, f(1) = 0, f(n) = f(n-1) + f(n-2)$.

5.3 Sums

Define for $n \in \mathbb{Z}_{\geq 0}, f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$.

$$\sum_{i=0}^{n} f(i)$$

Remark. *A few rules*

1. *"i" is a dummy variable so we can use anything.*
2. $$\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$$

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6.1 Supremum

Recall: Both $\mathbb{Q}$ and $\mathbb{R}$ should be ordered fields (satisfy axioms 1-10)
e.g. $\sqrt{2} \in \mathbb{R}, \notin \mathbb{Q}$

Definition. *A subset $S \subseteq \mathbb{R}$ is* <u>*bounded above*</u> *if $\exists y \in \mathbb{R}$ such that $\forall x \in S, x \leq y$.*

e.g. $S = \{3, 3.1, 3.14, 3.141, ...\}$ is bounded above by 4 (or π).

Definition. *An upper bound y for $S \subseteq \mathbb{R}$ is a least upper bound or supremum if y is an upper bound for S and if z is an upper bound for S, then $y \leq z$.*

11. Axiom(completeness): every nonempty bounded above set $S \subseteq \mathbb{R}$ has a supremum.

Proposition. *If y and y' are suprema of S, then $y = y'$. (suprema are unique)*

Proof. y is a supremum, and y' is an upper bound $\Rightarrow y \leq y'$ by definition of supremum. y' is a supremum, and y is an upper bound $\Rightarrow y' \leq y$ by definition of supremum. So $y = y'$ by trichotomy. □

Notation: $\sup S$ is the supremum of S.

6.2 Infimum

Definition. *S is bounded below if existsy $\mid \forall x \in S, y \leq x.y$ is a lower bound for S if $\forall x \in S, y \leq x$. y is the greatest lower bound or* <u>*infimum*</u> *if y is a lower bound and if z is any other lower bound, then $x \leq y$. $\overline{\inf S = \text{infimum}}$ of S.*

Definition. *If $S \subseteq \mathbb{R}$ is nonempty and bdd below, then $\exists!$ infimum $\inf S$.*

Proof. Let $-S = \{x \mid x \in S\}$. $-S$ is nonempty and bounded above. Thus, $\sup(-S)$ exists and is unique.
Claim: $-\sup(-S)$ is an infimum for S. Uniqueness same as for sup. □

Theorem 6.1 (Approximation Property)**.** *Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. $\forall \epsilon > 0, \exists x \in S \mid \sup S - \epsilon \leq x$.*

Proof by Contradiction. Assume $\exists \epsilon > 9. \forall x \in S, \sup S - \epsilon > x$. Then, $\sup S - \epsilon$ is an upper bound for S. Then, by definition of $\sup S, \sup S \leq \sup S - \epsilon$. Contradiction of $\epsilon > 0$. □

Theorem 6.2 (Additivity of Supremum)**.** *If $S, T \subseteq \mathbb{R}$ nonempty, bounded above, let $S + T := \{s + t \mid s \in S, t \in T\}$. Then $\sup(S + T)$ exists and equals $\sup(S) + \sup(T)$.*

Proof. Let $s = \sup S, t = \sup T$. $\forall x \in S, x \leq s, \forall y \in T, y \leq t \Rightarrow \forall x \in S, \forall y \in T, x + y \leq s + t \Rightarrow s + t$ is an upper bound for $S + T$. We want to show that $s + t$ is the <u>least</u> upper bound. Suppose not. Then $\exists \delta > 0 \mid s + t - \delta$ is an upper bound for $S + T$. Let $\epsilon = \frac{\delta}{2}$.

$$\begin{aligned} \text{Approximation Property} \Longrightarrow & \exists x \in S \mid s - \epsilon < x \\ & \exists y \in T \mid t - \epsilon < y \end{aligned}$$

Then $x + y \in S + T$. $s + t - \delta = s + t - 2\epsilon < x + y$. Contradiction! □

Proposition. *Suppose* S*,* $T \subseteq \mathbb{R}$ *such that* $\forall x \in S, \forall y \in T, x \leq y$*. Then* $\sup S$ *exists and* $\inf T$ *exists, and* $\sup S \leq \inf T$*.*

Proof. Any $x \in S$ is a lower bound for T
$\Rightarrow \inf T$ exists
Any $y \in T$ is upper bd for S
$\Rightarrow \sup S$ exists
Suppose $\sup S > \inf T$. Then $S - \sup S - \inf T$. Let $\epsilon = \frac{\delta}{2}$.
Approx $\Rightarrow \exists x \in S \mid \sup S - \epsilon < x$. Similar approx. result $\Rightarrow \exists y \in T \mid \inf T + \epsilon > y$.

$$y < \inf T + \epsilon = \sup S - \epsilon < x$$

Contradiction! $\sup S \leq \inf T$ □

Theorem 6.3. $\mathbb{Z}_{\geq 0} \subseteq \mathbb{R}$ *has no upper bound.*

Proof. By contradiction. If it did, let $\Psi = \sup \mathbb{Z}_{\geq 0}$. Approx w/ $\epsilon = \frac{1}{2} \Rightarrow \exists n \in \mathbb{Z}_{\geq 0}$ such that $\Psi - \frac{1}{2} < n$. Then $n + 1 \in \mathbb{Z}_{\geq 0}$ and $n + 1 > \Psi + \frac{1}{2} > \Psi$. Contradiction. □

6.3 Absolute Value

$|\cdot| : \mathbb{R} \to \mathbb{R}$
$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ "Distance" between $x, y \in \mathbb{R}$ is $|x - y|$. $|x - y| < \epsilon$ means x, y are "ϵ-close".

Theorem 6.4 (Triangle Inequality)**.**

$$|x + y| \leq |x| + |y|$$

$$|x - z| \leq |x - y| + |y - z|$$

Proof. Easy if $x = 0$ or $y = 0$. If both > 0, then LHS $= x + y =$ RHS. If both < 0, then LHS $= -x - y =$ RHS. If $y < 0 < x$, RHS $= x - y$

$$\left.\begin{array}{l} x - y > -x - y \\ x - y > x + y \end{array}\right\} \Longrightarrow x - y \geq |x + y|$$

Similar if $x < 0 < y$. □

7 September 25, 2019

7.1 Archimedean Prop.

Proposition (Archimedean Property)**.** *Let* $x > 0$ *and* $y \in \mathbb{R}$*. Then* $\exists n \in \mathbb{Z} \mid nx > y$*.*

Proof. Consider $\frac{y}{x}$. From earlier result, $\exists n \in \mathbb{Z}_{\geq 0}$ with

$$n > \frac{y}{x} \iff nx > y$$

□

Corollary 7.0.1. *If $a, x, y \in \mathbb{R}$ with $a \leq x \leq a + \frac{y}{n}$ for every $n \in \mathbb{Z}_{\geq 0}$, then $a = x$.*

Proof. Assume otherwise, so $a < x \iff x - a > 0$.

$$\begin{aligned} \text{Prop} \Rightarrow \exists n \in \mathbb{Z}_{\geq 0} \text{ such that } n(x-a) &> y \\ x - a &> \frac{y}{n} \\ x &> \frac{y}{n} + a \end{aligned}$$

□

7.2 Finite Sets

Notation $[n] = \{m \in \mathbb{Z}_{\geq 0} \mid 0 < m \leq n\}$
$= \{1, 2, 3, ..., n\}$

Theorem 7.1. $\forall m, n \in \mathbb{Z}_{\geq 0}$

1. *$\exists$ inj. $f : [m] \to [n] \iff m \leq n$*
2. *$\exists$ surj. $f : [m] \to [n] \iff m \geq n$*

Proof. Proof of both.

1. ($\Leftarrow$) $m \leq n$ so define $f : [m] \to [n]$. $f(i) = i$ so f is clearly injective.

   ($\Rightarrow$) Induct on m (fix n). Assume for given m, all n. Suppose $f : [m+1] \to [n]$ is injective. $\forall i \in [m+1]$, if $i \neq m+1 \Rightarrow f(m+1)$. Define $\bar{f} : [m] \to [n] \backslash \{f(m+1)\}$. $\bar{f}(i) = f(i)$ so it is stilll injective. Define $h : [n] \backslash \{f(m+1)\} \to [n-1]$.
   $h(i) = \begin{cases} i \text{ if } i < f(m+1) \\ i - 1 \text{ if } i > f(m+1) \end{cases}$ Easy to show that h is injective. Then $h \circ \bar{f} : [m] \to [n-1]$ is injective (composition of injective functions). Inductive hypothesis $\Rightarrow m \leq n - 1 \Rightarrow m + 1 \leq n$

2. Similar to 1

□

Definition. *A set is finite if $\exists$ bijection $f : [n] \to S$ for some $n \in \mathbb{Z}_{\geq 0}$. If not, S is <u>infinite</u>.*

Proposition. *Given finite set S, the $n \in \mathbb{Z}_{\geq 0}$ as above is <u>unique</u>.*

Proof. Suppose $f : [n] \to S.g : [m] \to S$ are bijections. Then let $h : S \to [n]$ be inverse of f. Then $h \circ g : [m] \to [n]$ is bijective.

$$\begin{aligned} \text{Thm applied to } h \circ g \Longrightarrow & m \leq n \\ & n \leq m \\ \Longrightarrow & n = m \end{aligned}$$

□

<u>Ex</u>
\# of elemenets: $|\dot{S}|$ (like magnitude). If S, T are finite sets, then $S \cup T, S \cap T, S \times T, S^T$ are finite. Any subset of S is finite. If $S \subseteq T, |\dot{S}| \leq |\dot{T}|$.

<u>Ex</u>
$\mathbb{Z}_{\geq 0}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are infinite.

Notation: Suppose $f : S \to T, U \subseteq S$. Then, $f|_U : U \to T.f|_U(s) = f(s), s \in U$.

Definition. *The inclusion of $S \subseteq T$ is $id_T|_S$.*

<u>Ex</u>
f injective $\Rightarrow f|_U$ injective. Does <u>not</u> hold for sujectivity.

Definition. *For $a \leq b \in \mathbb{R}$, an <u>interval</u> is one of*

$$\begin{aligned} [a, b] &= \{a \leq x \leq b \mid x \in \mathbb{R}\} \\ (a, b) &= \{a < x < b \mid x \in \mathbb{R}\} \\ [a, b) &= \{a \leq x < b \mid x \in \mathbb{R}\} \\ (a, b] &= \{a < x \leq b \mid x \in \mathbb{R}\} \end{aligned}$$

<u>or</u> allow a or b to be "∞" or "$-\infty$" for open intervals w/ $-\infty < x < \infty \forall x \in \mathbb{R}$.

<u>Ex</u>
If $a \neq b$, these are infinite sets.

Definition. *Some function definitions*

$$\begin{array}{lll} \text{If } f : [a, b] \to \mathbb{R} & \text{then} & f + g : [a, b] \to \mathbb{R} \\ \quad g : [a, b] \to \mathbb{R} & & fg : [a, b] \to \mathbb{R} \end{array}$$

If $c \in \mathbb{R}, cf : [a, b] \to \mathbb{R}$. $(cf)(x) = cf(x)$.

Definition. *If* $f_1, \dots, f_n : [a, b] \to \mathbb{R}, c_1, \dots, c_n \in \mathbb{R}$, *then the corresponding linear combination is*

$$\sum_{i=1}^{n} c_i f_i = c_1 f_1 + c_2 f_2 + \dots c_n f_n$$

7.3 Step Functions

Definition. $f : [a, b] \to \mathbb{R}$ *is a step function if* $\exists$ *a finite set of real numbers* $S = \{x_0, \dots, x_n\} \subset \mathbb{R}$, *called a partition for* f, *with* $a = x_0 < x_1 < x_2 < \dots < x_n = b$ *and* $c_1 \dots c_2 \in \mathbb{R}$ *such that* $\forall i \in [n], \forall x \in (x_{i-1}, x_i), f(x) = c_i$.

$$(i.e\, f|_{(x_{i-1}, x_i)} = c_i)$$

Proposition. *If* $f, g : [a, b] \to \mathbb{R}$, *so are* $f + g, fg$.

Proof. Let S be partition for f and let T be partition for g. Idea: Then $S \cup T$ is a partition for $f + g$ and fg. Let $S = \{x_0, \dots, x_m\}, T = \{y_o, \dots, y_n\}$. $S \cup T = \{z_0, \dots, z_p\}$ (is finite by earlier exercise). For any $z_k \in S \cup T$ with $k > 0$, let k_i = greatest element of $S \mid x_i < z_k$. Let y_j = greatest element of $T \mid y_j < z_k$. Then z_{k-1} = a maximum of x_i, y_j and z_k = a minimum of x_{i+1}, y_{j+1}.
Hence $(z_{k-1}, z_k) \subset (x_i, x_{i+1}) \cap (y_j, y_{j+1})$
$\Rightarrow f + g, fg$, constant on (z_{k-1}, z_k) □

8 September 30, 2019

8.1 Integrals of Step Functions

Definition (Integral of step functions)**.** *Let* f *be a step function on* $[a, b]$ *with partition* $\{x_0, x_1, \dots, x_n\}$ *and such that* $f|_{(x_{i-1}, x_i)}(x) = c_i$ *with* $c_1 \in \mathbb{R}$. *Then*

$$\int_a^b f = \sum_{i-1}^{n} c_i (x_i - x_{i-1})$$

Proposition. *This is well-defined (doesn't depend on partition).*

Proof. Given two partitions P, Q

1. Assume $P \leq Q$. Sketch: use that $c(x_{i+1} - x_i) = c(x_{i+1} - x_{i+1}) - c(x_{i+1} - x_i)$ and use induction on $|Q - P|$.

2. In general, for any P, Q we have $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$ so reduces to 1 twice.

□

<u>Conventions</u>

$$\int_a^a f = 0$$

$$\text{if } b < a, \int_a^b f = -\int_b^a f$$

Theorem 8.1 (Properties of $\int$ for step functions)**.** *$f, g : [a, b] \to \mathbb{R}$ and $c, c_i \in \mathbb{R}$*

1.
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

2.
$$\int_a^b cf = c\int_a^b f$$

3.
$$\left(\sum_{i=1}^n c_i f_i\right) = \sum_{i=1}^n c_i \int_a^b f_i$$

4. *If $f \leq g$ (i.e $\forall x \in [a, b], f(x) \leq g(x)$), then*
$$\int_a^b f \leq \int_a^b g$$

5.
$$\int_a^c f = \int_a^b f + \int_b^c f$$

6.
$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$$

7. *If $c \neq 0$, then*
$$\int_{ca}^{cb} f\left(\frac{x}{c}\right) dx = c\int_a^b f(x)dx$$

Proof of additivity. If P, Q are partitions for f, g, then $P \cup Q$ is partition for f and g. Say $f|_{(x_{i-1},x_i)} c_i$ and $g|_{(x_{i-1},x_i)} d_i$. $(f+g)|_{(x_{i-1},x_i)}(x) = c_i + d_i$. So

$$\begin{aligned}
\int_a^b (f+g) &= \sum_{i=1}^n (c_i + d_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n (c_i(x_i - x_{i-1}) + d_i(x_i - x_{i-1})) \\
&= \sum_{i=1}^n c_i(x_i - x_{i-1}) + \sum_{i=1}^n d_i(x_i - x_{i-1}) \\
&= \int_a^b f + \int_a^b g
\end{aligned}$$

Lemma 8.2. $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Pf of Lemma. Induct on n. Base case: $n = 0\checkmark$. Inductive Step: Assume for n.

$$\begin{aligned}\sum_{i=1}^{n+1}(a_i + b_i) &= \sum_{i=1}^{n}(a_i + b_i) + a_{n+1} + b_{n+1} \\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i + a_{n+1} + b_{n+1} \\ &= \sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n+1} b_i\end{aligned}$$

□

□

8.2 More General Functions

Let $f : [a, b] \to \mathbb{R}$ be a function.

Definition. *f is <u>bounded</u> (on $[a, b]$) if $\exists c \in \mathbb{R}$ such that $|f(x)| \leq c$ for all $x \in [a, b]$.*

Definition. *For f bounded, the lower integral is $\underline{I}(f) = \sup \underline{S}(f)$ where $\underline{S}(f) = \left\{\int_a^b s \mid s \text{ is a step function and } s \leq f\right\}$. Similarly, let $\bar{I}(f) = \inf \bar{S}(f)$(upper integral)where $\bar{S}(f) = \left\{\int_a^b t \mid t \text{ is a step function and } f \leq t\right\}$.*

Proposition. *This is well defined.*

Proof. For $\underline{I}(f)$, need to check that $\underline{S}(f)$ is not empty and bounded above. Constant function $s(x) = -C$ is a step function and $s \leq f$ so $\underline{S}(f)$ is nonempty. Let $t(x) = C$. Then $f \leq t$. So any step function s with $s \leq f$ satifies $s \leq t$. By comparison for step functions,

$$\int_a^b s \leq \int_a^b t$$

so this number is an upper bound for $\underline{S}(f)$. Similar for upper integral. □

Definition. *If $f : [a, b] \to \mathbb{R}$ is a bounded function, then we say f is integrable if $\underline{I}(f) = \bar{I}(f)$ and the integral is*

$$\int_a^b f = \underline{I}(f) = \bar{I}(f)$$

Proposition. $\underline{I}(f) \leq \bar{I}(f)$

Proof. $\forall s, t$ with $s \leq f \leq t$, we have $\int_a^b s \leq \int_a^b t$ by comparison for step functions. So $\forall x \in \underline{S}(f), \forall y \in \bar{S}(f), x \leq y \Rightarrow \sup \underline{S}(f) \leq \inf \bar{S}(f)$. □

Theorem 8.3. *Let $f : [0,1] \to \mathbb{R}$ be defined $f(x) = \begin{cases} 1 \; if \, x \in \mathbb{Q} \\ 0 \; if \, x \notin \mathbb{Q} \end{cases}$. We will assume there exists some $x \in \mathbb{Q}$.*

Proof. Homework Problem $\Longrightarrow \forall a < b \in \mathbb{R}, \exists x$ with $a < x < b$ and $x \in \mathbb{Q}$. There is some irrational $x \in \mathbb{R}$. Arch. Prop $\Rightarrow \exists n \in \mathbb{Z}_{\geq 0}$ with $n > x$. Thus,

$$\begin{array}{ccccc} 0 & < & x & < & n \\ 0 & < & \frac{x}{n} & < & 1 \\ 0 & < & (b-a)\frac{x}{n} & < & b-a \\ a & < & (b-a)\frac{x}{n}+a & < & b \end{array}$$

Sums, products, quotients, differences of rational numbers are rational, so if $(b-a)\frac{x}{n}+a$ were rational then $\left(\frac{(b-a)\frac{x}{n}+a-a}{b-a}\right)n = x \Rightarrow x$ is rational. Thus, $(b-a)\frac{x}{n}+a$ is rational by contradiction. □

Lemma 8.4. *Now suppose $a < b$ are arbitrary reals. HW Prob $\Rightarrow \exists c, d \in \mathbb{Q}$ with $a < c < d < b$. Apply previous to $c, d \Rightarrow \exists$ irrational $x, c < x < d \Rightarrow a < x < b$. "$\mathbb{R} \backslash \mathbb{Q}$ is dense in $\mathbb{R}$".*

9 October 2, 2019

9.1 Integrable Functions

$f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$. We know: $\forall a, b, \exists x \in \mathbb{Q} \mid a < x < b$ and $\exists y \notin \mathbb{Q} \mid a < y < b$.

Proof that f is not integrable. Let s be step function with $s \leq f$. $\exists$ partition $P = \{x_0, \ldots, x_n\}$ with $s|_{(x_{i-1}, x_i)}$ constant.
$\exists y \notin \mathbb{Q}$ with $x_{i-1} < y < x_i$
$\Rightarrow f(y) = 0$
$\Rightarrow s(y) \leq 0$
$\Rightarrow s|_{(x_{i-1}, x_i)} \leq 0$
$\Rightarrow s \leq 0$ except at points of P (maybe)
Hence $\int_0^1 s \leq 0 \Rightarrow \underline{I}(F) \leq 0$ (in fact $\underline{I}(f) = 0$ since $s = 0$ works). Similarly, if

$t \geq f$, then $t|_{(x_{i-1},x_i)}$ = constant on subints of same partition.

$$\begin{aligned}\exists x \in \mathbb{Q} \mid x_{i-1} < x < x_i &\Rightarrow f(x) = 1 \\ &\Rightarrow t(x) \geq 1 \\ &\Rightarrow t|_{(x_{i-1},x_i)} \\ &\Rightarrow t \geq 1 \text{(except maybe at pts of partition)} \\ &\Rightarrow \int_0^1 t \geq 1 \\ &\Rightarrow \bar{I}(f) \geq 1\end{aligned}$$

In fact $\bar{I}(f) = 1$ because $t = 1$ works. Thus, $\underline{I}(f) \neq \bar{I}(f) \Rightarrow f$ not integrable. □

Theorem 9.1. *All previous properties of $\int_a^b$ for step functions also hold for integrable functions. (also if functions f, g, f_i are integrable, so are $f + g$, etc ...)*

Pf of Additivity. Assuming f, g are integrable.

Lemma 9.2. *If $S \subseteq T$ bdd above, nonempty in $\mathbb{R}$, then $\sup S \leq \sup T$.*

Proof. $\sup T$ is an upper bd for S, so $\sup S \leq \sup T$. □

Let $s \leq f, s' \leq g$ be step functions $\Rightarrow s + s' \leq f + g$. Additivity for step functions $\Rightarrow \int s + s' = \int s + \int s' \Rightarrow$ if $x \in \underline{S}(f)$ and $y \in \underline{S}(g)$, then $x + y \in \underline{S}(f + g)$. i.e. $\{x + y \mid x \in \underline{S}(f), y \in \underline{S}(g)\} \subseteq \underline{S}(f + g)$.
Lemma $\Rightarrow \sup\{x + y \mid x \in \underline{S}(f), y \in \underline{S}(g)\} \leq \underline{I}(f + g)$.
Additivity of sup $\Rightarrow \underline{I}(f) + \underline{I}(g) \leq \underline{I}(f + g)$. Similarly, we get $\bar{I}(f) + \bar{I}(g) \geq \bar{I}(f + g)$.

$$\begin{aligned}f, g \text{ integrable} \Longleftrightarrow & \underline{I}(F) = \bar{I}(f) \\ & \underline{I}(g) = \bar{I}(g)\end{aligned}$$

$$\int f + \int g = \underline{I}(f) + \underline{I}(g) \leq \underline{I}(f + g) \leq \bar{I}(f + g) \leq \bar{I}(f) + \bar{I}(g)$$

This implies that all $\leq$ are $=$. Thus we get the result. □

9.2 Monotonicity

Definition. *Say $f : [a, b] \to \mathbb{R}$ is <u>nonincreasing</u> if $\forall x, y \in [a, b], x \leq y \Rightarrow f(x) \geq f(y)$ or is <u>nondecreasing</u> if $\forall x, y \in [a, b], x \leq y \Rightarrow f(x) \leq f(y)$. A function is monotonic if it is nonincreasing or nondecreaisng.*

Theorem 9.3. *If $f : [a, b] \to \mathbb{R}$ is monotonic, then it is integrable.*

Lemma 9.4. *If f is monotonic on $[a, b]$, it is bounded.*

Proof. Suppose f nondecreasing. Let $C = max\{|f(a)|, |f(b)|\}$. If $x \in [a, b]$,

1. if $f(x) \geq 0$, then $|f(x)| = f(x) \leq f(b) = |f(b)| \leq C$.
2. if $f(x) < 0$, then $|f(x)| = -f(x) \leq -f(a) \leq |f(a)| \leq C$.

f nonincreasing similar. □

Proof of Theorem. Important to note, f nonincr. $\iff -f$ non decreasing. By properties of $\int$, f integrable $\iff -f$ integrable. So suffices to show when f is nondecreasing can also assume $[a,b] = [0,1]$: indeed, let $g(x) = f(a+(b-a)x)$. $g : [0,1] \to \mathbb{R} \Rightarrow f(y) = g\left(\frac{1}{b-a}y + \frac{a}{b-a}\right)$. By translation + dilation properties f int $\iff g$ int. Pick $n \in \mathbb{Z}_{>0}$. $s_n, tn : [0,1] \to \mathbb{R}. s_n(x) = f\left(\frac{\lfloor nx \rfloor}{n}\right)$.

$t_n(x) = \begin{cases} f\left(\frac{\lfloor nx \rfloor + 1}{n}\right) \text{ if } x < 1 \\ f(1) \text{ if } x = 1 \end{cases}$

$\Rightarrow s_n, t_n$ are step functions as $s_n \mid_{\left(\frac{i-1}{n}, \frac{i}{n}\right)} = f\left(\frac{i-1}{n}\right)$ and $t_n \mid_{\left(\frac{i-1}{n}, \frac{i}{n}\right)} = f\left(\frac{i}{n}\right)$

Also, $s_n \leq f_n \leq t_n$ by same description and nondecreasingness. Hence,

$$\int_0^1 s_n \leq \underline{I}(f) \leq \bar{I}(f) \leq \int_0^1 t_n$$

$$0 \leq \bar{I}(f) - \underline{I}(f) \leq \int_0^1 (t_n - s_n) \text{ (using additivity)}$$

$$\begin{aligned} \int_0^1 (t_n - s_n) &= \sum_{i=1}^{n} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right) \frac{1}{n} \\ &= \frac{1}{n}(f(1) - f(0)) \end{aligned}$$

Therefore, $\forall n \in \mathbb{Z}_{\geq 0}, 0 \leq \bar{I}(f) - \underline{I}(f) \leq \frac{1}{n}(f(1) - f(0))$. $\forall n, n \leq \frac{f(1)-f(0)}{\bar{I}(f) - \underline{I}(f)}$. Violates arch. property $\Rightarrow \bar{I}(f) - \underline{I}(f) = 0 \Rightarrow f$ integrable. □

10 October 7, 2019

10.1 Piecewise Monotonicity

Recall: We know that monotonic $\Rightarrow$ integrable.

Definition. *$f : [a,b] \to \mathbb{R}$ is piecewise monotonic if $\exists$ partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that $f \mid_{(x_{i-1}, x_i)}$ is monotonic for $1 \leq i \leq n$.*

Corollary 10.0.1. *If $f : [a,b] \to \mathbb{R}$ is piecewise monotonic, then f is integrable.*

Proof. Concatenation + Induction □

Corollary 10.0.2. *f a linear combo of piecewise monotonic functions $\Rightarrow f$ integrable.*

Proof. Use linearity of $\int$ □

Corollary 10.0.3. *The function* $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ *on* $[0,1]$ *is* *not* *a linear combo of piecewise monotonic functions*

Proof. Nothing was written? □

10.2 Polynomials

Definition. *For* $x \in \mathbb{R}, n \in \mathbb{Z}_{\geq 0}$*, defined inductively* $x^0 = 1$ *and* $x^{n+1} = x^n \cdot x$*. A polynomial is a linear combo of* $x \mapsto x^n$ *i.e.* $f(x) = \sum_{n=1}^{N} c_n x^n$*.*

Corollary 10.0.4. *Polynomials on* $[a,b]$ *are integrable.*

Proof. HW □

Proposition. *If* $a \leq c \leq d \leq b \in \mathbb{R}$ *and* $f[a,b] \to \mathbb{R}$ *integrable on* $[a,b]$*, then* $f\mid_{[c,d]}$ *is integrable.*

Proof. On HW? □

Definition. *If* $f : [a,b] \to \mathbb{R}$ *is integrable, let the indefinite integral of* f *be the function* $g(x) = \int_a^x f$ $(g : [a,b] \to \mathbb{R})$*.*

10.3 Limits

Definition. *For* $f : [a,b] \to \mathbb{R}, c \in [a,b], k \in \mathbb{R}$*, say* $\lim_{x \to c} f(x) = K$ *whenever* $\forall \epsilon > 0, \exists \delta > 0$ *such that* $\forall x \in [a,b], 0 < |x - c| < \delta \Rightarrow |f(x) - K| < \epsilon$*.*

Ex

1. $f(x) = K, \lim_{x \to c} f(x) = K$. For any $\epsilon > 0$, pick $\delta = 13\frac{2}{3}$. Then $0 < |x - c| < \delta \Rightarrow |f(x) - K| = 0 < \epsilon$.

2. $f(x) = x, \lim_{x \to c} f(x) = c$. For any $\epsilon > 0$, pick $\delta = \epsilon$. Then $0 < |x - c| < \delta = \epsilon \Rightarrow 0 < |f(x) - c| < \epsilon$.

3. $f(x) = ax, a \in \mathbb{R}, \lim_{x \to c} f(x) = ac$. For any $\epsilon > 0$, pick $\delta = \frac{\epsilon}{|a|}$. Then $0 < |x - c| < \delta \Rightarrow 0 < |ax - ac| < \delta|a| = \epsilon$.

4. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Claim: There is no K for which $\lim_{x \to 0} f(x) = K$. We say the limit "does not exist".

Proof for Claim in 4. Suppose it did, so $\lim_{x \to 0} f(x) = K$. Let $\epsilon = \frac{1}{2} > 0$. $\exists \delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow |f(x) - K| < \frac{1}{2}$. Choose $x_0 \in \mathbb{Q}$ such that $0 < |x_0| < \delta$. Choose $x_1 \notin \mathbb{Q}$ such that $0 < |x_1| < \delta$. Then $|f(x_0) - K| < \frac{1}{2} \Rightarrow |1 - K| < \frac{1}{2} \Rightarrow -\frac{1}{2} < 1 - K < \frac{1}{2} \Rightarrow \frac{1}{2} < K < \frac{3}{2}$. Also, $|f(x_1) - K| < \frac{1}{2} \Rightarrow |K| < \frac{1}{2} \Rightarrow -\frac{1}{2} < K < \frac{1}{2}$ which is a contradiction. □

Proposition. $\lim_{x\to c} f(x)$ *is unique if it exists.*

Proof. Assume $\lim_{x\to c} f(x) = K_1$ and $\lim_{x\to c} f(x) = K_2$. $\forall \epsilon, \exists \delta_1 > 0 \mid - < |x-c| < \delta_1 \Rightarrow |f(x)-K_1| < \epsilon$ and $\exists \delta_2 > 0 \mid 0 < |x-c| < \delta_2 \Rightarrow |f(x)-K_2| < \epsilon$. Choose x with $0 < |x-\epsilon| < \min(\delta_1, \delta_2)$. Then

$$\begin{aligned} |K_1 - K_2| &= |K_1 - f(x) + f(x) - K_2| \\ &\leq |K_1 - f(x)| + |f(x) - K_2| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \Rightarrow K_1 = K_2 \end{aligned}$$

□

Theorem 10.1 (Basic limit Theorem)**.** *If* $\lim_{x\to c} f(x) = K, \lim_{x\to c} g(x) = L$, *then*

1. $\lim_{x\to c}(f(x) + g(x)) = K + L$
2. $\lim_{x\to c}(f(x) - g(x)) = K - L$
3. $\lim_{x\to c}(f(x) \cdot g(x)) = K \cdot L$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{K}{L}$ *if* $L \neq 0$ *and* $g(x) \neq 0$ *for all* x.
5. *Linearity:*

$$\lim_{x\to c} \sum_{i=1}^{N} f_i(x) = \sum_{i=1}^{N} \lim_{x\to c} f_i(x)$$

Proof.

1. For $\epsilon > 0, \exists \delta_1, \delta_2 \mid 0 < |x-c| < \delta_1 \Rightarrow |f(x) - K| > \frac{\epsilon}{2}$ and $0 < |x-c| < \delta_2 \Rightarrow |g(x) - L| < \frac{\epsilon}{2}$. Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x-c| < \delta \Rightarrow$

$$|f(x) + g(x) - K - L| \leq |f(x) - K| + |g(x) - L| < \epsilon$$

□

3. Assume first that $L = 0$. Let $D = \max\{1, |K|\} > 0$. Know $\forall \epsilon_1 > 0, \exists \delta_1 > 0 \mid - < |x-c| < \delta_1 \Rightarrow |f(x) - K| < \epsilon_1$. In particular, thsi is true for $\epsilon_1 = D$. Then

$$\begin{aligned} 0 < |x-c| < \delta_1 \Rightarrow |f(x)| &= |f(x) - K + K| \\ &\leq |f(x) - K| + |K| \\ &\leq 2D \end{aligned}$$

Also know $\forall \epsilon_2 > 0, \exists \delta_2 > 0 \mid 0 < |x-c| < \delta_2 \Rightarrow |g(x)| < \epsilon_2$. In particular, for any $\epsilon > 0$, pick $\epsilon_2 = \frac{\epsilon}{2D}$. Take $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x-c| < \delta \Rightarrow |f(x)g(x) - KL| = |f(x)g(x)| < (2D)\frac{\epsilon}{2D} = \epsilon$.

□

11 October 9, 2019

11.1 Limits Cont.

Recall: We were trying to prove if $\lim_{x\to c} f(x) = K, \lim_{x\to c} g(x) = L$, then $\lim)x \to cf(x)g(x) = KL$ and $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{K}{L}$ if $g(x) \neq 0$ and $L \neq 0$.

Proof. We showed $\lim_{x\to c} f(x)g(x) = 0$ if $L = 0$. In general, $f(x)g(x) - KL = f(x)(g(x) - L) + (f(x) - K)(L)$.

$$\begin{aligned}\lim_{x\to c}(f(x)g(x) - KL) &= \lim_{x\to c}(f(x)(g(x) - L) + ((f(x) - K)(L)) \\ &= \lim_{x\to c} f(x)(g(x) - L) + \lim_{x\to c}(f(x) - K)(L) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to c} f(x)g(x) &= \lim_{x\to c}(f(x)g(x) - KL + KL) \\ &= \lim_{x\to c} f(x)g(x) - \lim_{x\to c}(KL + KL) \\ &= \lim_{x\to c}(f(x)g(x) - KL) + KL \\ &= 0 + KL \\ &= KL\end{aligned}$$

Now lets show division. We know

$$\begin{aligned}\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} f(x) \cdot \lim_{x\to c} \frac{1}{g(x)} \\ &= K \cdot \lim_{x\to c} \frac{1}{g(x)}\end{aligned}$$

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| = \left|\frac{L - g(x)}{g(x)L}\right|$$

Note that $|g(x) - L| < \frac{|L|}{2}$.

$$\begin{aligned}|L| &= |L + g(x) - g(x)| \\ &\leq |L - g(x)| + |g(x)| \\ &< \frac{|L|}{2} + |g(x)| \\ &\Rightarrow \frac{|L|}{2} < g(x)\end{aligned}$$

Given $\epsilon > 0$, take $\epsilon_1 = \min(\frac{|L|}{2}, \frac{|L|^2}{2}\epsilon)$. Then, $\exists \delta_1 > 0 \mid 0 < |x - c| < \delta_1 \Rightarrow |g(x) - L| < \epsilon_1$. $\frac{|L|}{2} < g(x) \Rightarrow |g(x)| > \frac{|L|}{2}$.

$$
\begin{aligned}
\left|\frac{1}{g(x)} - \frac{1}{L}\right| &= \left|\frac{L-g(x)}{g(x)L}\right| &&= \frac{|L-g(x)|}{|g(x)||L|} \\
&< &&< \frac{2|L-g(x)|}{|L|^2} \\
& &&< \frac{2}{|L|^2} \cdot \frac{|L|^2}{2}\epsilon
\end{aligned}
$$

Thus, $\left|\frac{1}{g(x)} - \frac{1}{L}\right| < \epsilon$. □

Theorem 11.1 (Squeeze Theorem)**.** *If $f \le g \le h$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = K$ exists then $\lim_{x\to c} g(x) = K$.*

Proof. Given $\epsilon > 0, \exists \delta_1 > 0 \mid 0 < |x-c| < \delta_1 \Rightarrow |f(x) - K| < \epsilon$ and $\exists \delta_2 > 0 \mid 0 < |x-c| < \delta_2 \Rightarrow |h(x) - K| < \epsilon$. Pick $\delta = \min(\delta_1, \delta_2)$. Then,

$$
\begin{aligned}
0 < |x-c| < \delta &\Rightarrow K - \epsilon \le f(x) \le g(x) \le h(x) \le K + \epsilon \\
&\Rightarrow K - \epsilon < g(x) < K + \epsilon \\
&\Leftrightarrow |g(x) - K| < \epsilon
\end{aligned}
$$

□

11.2 Continuity

Definition. *$f : [a,b] \to \mathbb{R}$ is continuous at $c \in [a,b]$ if $\lim_{x\to c} f(x) = f(c)$. Discontinuous if not. f is continuous if it is continuous at each $c \in [a,b]$.*

Ex $f(x) = K$
We showed $\lim_{x\to 0} f(x) = K \Rightarrow f$ is continuous

Ex $f(x) = ax$
We showed $\lim_{x\to c} f(x) = \lim_{x\to c}(ax) = ac = f(c) \Rightarrow f$ is continuous

Ex $f(x) = ax^n$
Induction + limit thm for result $\Rightarrow \lim_{x\to c} f(x) = ac^n = f(c) \Rightarrow f$ is continuous

Ex Polynomials are continuous

Ex $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$
Calculate $\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$ so f is not continuous at $x = 0$.

Ex $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$
We showed $\lim_{x\to 0} f(x)$ does not exists $\Rightarrow f$ is not continuous.

Theorem 11.2. *If f, g are continuous at c, then*

1. *$f + g$ is cont. at c*

2. *$f - g$ is cont. at c*

3. *fg is cont. at c*

4. *f/g is cont. at c if $g(x) \neq 0$ for all x*

"Pf".

1. We know $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. By adding these we get

$$\lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$$
$$\lim_{x \to c} (f(x) + g(x)) = (f + g)(c)$$

□

Facts(easy):

- f is cont. at $c \iff \forall \epsilon > 0, \exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$
- $\lim_{x \to c} f(x) = K \iff \lim_{h \to 0} f(c + h) = K$

Theorem 11.3. *If $f : [a, b] \to \mathbb{R}$ is continuous at $c \in [a, b]$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(c)$, then $g \circ f : [a, b] \to \mathbb{R}$ is continuous at c.*

Proof. We know (1) $\forall \epsilon > 0, \exists \delta_1 > 0 \mid |x - c| < \delta_1 \Rightarrow |f(x) - f(c)| < \epsilon_1$
(2) $\forall \epsilon_2 > 0, \exists \delta_2 > 0 \mid |y - f(c)| < \delta_2 \Rightarrow |g(y) - g(f(c))| < \epsilon_2$
Given $\epsilon > 0$. Take $\epsilon_2 = \epsilon$. From (2), get $\delta_2 > 0$. Take $\epsilon_1 = \delta_2$. From (1), get $\delta_1 > 0$. Choose $\delta = \delta_1$. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon_1$. Let $y = f(x)$.

$$\Rightarrow |g(f(x)) - g(f(c))| < \epsilon_2 = \epsilon$$
$$|(g \circ f)(x) - (g \circ g)(c)| < \epsilon$$

□

Theorem 11.4. *Let $f : [a, b]$ be continuous. Then f is bounded.*

<u>Ex</u>
Not true if open interval

$$f : (0, 1] \to \mathbb{R} \quad f(x) = \frac{1}{x}$$

Not true of discontinuous at only <u>one</u> point.

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$$

Not true if interval is not bounded

$$f : [0, \infty) \to \mathbb{R} \quad f(x) = x$$

12 October 14, 2019

12.1 Boundedness

Theorem 12.1. *A continuous function $f : [a,b] \to \mathbb{R}$ is bounded.*

Lemma 12.2. *Let $S = \{x \in [a,b] \mid f \text{ is bounded on} [a,x]\}$. Then $c = \sup S$ and $\exists d > c \mid f$ is bounded on $[a, \min(b,d)]$.*

Proof of Thm (assuming lemma). Let c, d be as in lemma. $S \subseteq [a,b] \Rightarrow c \leq b$. If $c < b$, then $c < \min(b.d)$ and Lemma $\Rightarrow f$ bdd on $[a, \min(b,d)]$. Therefore $\min(b.d) \in S > \sup S \Rightarrow\Leftarrow$ Contradiction. So $c = b$, so $\min(b,d) = b$. Lemma $\Rightarrow f$ bdd on $[a,b]$. □

Proof of Lemma. We know $a \in S, b$ upper bd for $S \Rightarrow c = \sup S$ exists. Take $\epsilon = 1$ in def of continuity at $x = c$. We then know that $-1 < f(x) - f(c) < 1 \Rightarrow f(c) - 1 < f(x) < f(c) + 1$. Let $k = \max(|f(c) - 1|, |f(c) + 1|)$. Therefore, $|f(x)| < K$. Hence, f is bdd on $(c - \delta, c + \delta)$. By approximation property of sup, $\exists y \in S \mid c - \delta < y \leq C \Rightarrow f$ bdd on $[a,y] \Rightarrow f$ is bdd on $[a, c + \delta]$. Let $d = c + \frac{\delta}{2} \Rightarrow f$ is bdd on $[a, \min(b,d)]$. □

12.2 Minimums and Maximums

Theorem 12.3 (Extreme Value Thm)**.** *A continuous $f : [a,b] \to \mathbb{R}$ has an absolute min/max.*

Proof. Let $M(f) = \sup\{f(x) \mid x \in [a,b]\}$ and $m(f) = \inf\{f(x) \mid x \in [a,b]\}$(existence by previous theorem). Suppose there is <u>no</u> $c \in [a,b] \mid f(c) = M(f)$. Then the function $M(f) - f(x)$ is nonzero $\Rightarrow g(x) = \frac{1}{M(f)-f(x)}$ is defined and continuous on $[a,b]$. Previous Thm $\Rightarrow$ it is bounded, say $|g(x)| \leq K$. Approx prop of sup $\Rightarrow \exists x \in [a,b] \mid 0 \leq M(f) - f(x) < \frac{1}{K}$. Then $|g(x)| = \frac{1}{|M(f)-f(x)|} > K$. Similar for $m(f)$. □

<u>An application</u>
Let $f : [a,b] \to [a,b]$ satisying $|f(x) - f(y) < |x - y|$ if $x \neq y$ (f is "contracting"). Then $\exists! c$ such that $f(c) = c$ (a "fixed point")

Proof. Uniqueness: if $f(c) = c, f(d) = d$, then $|f(c) - f(d)| = |c - d| \Rightarrow c = d$ contradiction
Existence: Let $g(x) = |f(x)0x|$. g is continuous by limit rules. Extreme Value Thm $\Rightarrow \exists c \mid \forall x \in [a,b], g(c) \leq g(x)$. Suppose $g(c) \neq 0$. $f(c) \neq 0$, so

$$|g(f(c))| = |f(f(c)) - f(c)| < |f(c) - c| = g(c) \text{ by definition}$$

$|g(f(c))| < |g(c)|$ so contradiction $\Rightarrow g(c) = 0 \Rightarrow f(c) = c$ □

Definition. *$f : [a,b] \to \mathbb{R}$, let span$(f) = M(f) - m(f)$.*

Theorem 12.4 (Small-Span Thm)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous, then $\forall \epsilon > 0, \exists$ a partition $P = \{x_0, \ldots, x_n\}$ at $[a,b]$ such that span$(f \mid_{[x_{i-1}, x_i]}) < \epsilon$.*

Lemma 12.5. *Fix $\epsilon > 0$. Let $S = \{x \mid$ the conclusion of Thm is true for $f_{[a,x]}\}$. Then $\exists c = \sup S$, and $\exists d > c$ such that thm is true on $[a, \min(b,d)]$.*

Lemma $\Rightarrow$ Thm exactly as before

Pf of Lemma. $a \in S, S$ bdd above by $b \Rightarrow c = \sup S$ exists. Continuity at $c \Rightarrow \exists \delta > 0 \mid |x - c| < \delta \Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2}$. We then can get the following inequality.

$$\frac{-\epsilon}{2} < f(x) - f(c) < \frac{\epsilon}{2} (*)$$

EVT applied to $f \mid_{[c-\frac{\delta}{2}, c+\frac{\delta}{2}] \cap [a,b]}$ has a max M, min m. By $(*), M < f(c) + \frac{\epsilon}{2}$ and $m > f(c) - \frac{\epsilon}{2}$.

$$M - m < \left(f(c) + \frac{\epsilon}{2}\right) - \left(f(c) - \frac{\epsilon}{2}\right) = \epsilon$$

Pick $d = c + \frac{\delta}{2}$. Approx $\Rightarrow \exists y \in S \mid y > c - \frac{\delta}{2}$. $\exists P = \{x_0 = a, \ldots, y\}$ such that thm holds. Let $P' = \{x_0, \ldots, x_n = y, \min(b,d)\}$. Then lemma holds with f' on $[a, \min(b,d)]$. □

13 October 16, 2019

13.1 Integrability

Theorem 13.1. *$f : [a,b] \to \mathbb{R}$ continuous. Boundedness $\Rightarrow f$ integrable.*

Proof. Bddness thm $\Rightarrow f$ bdd. Given $\epsilon > 0$, Small-Span Thm $\Rightarrow \exists P = \{x_0, \ldots, x_n\}$ such that $f \mid_{[x_{i-1},x_i]} < \frac{\epsilon}{b-a}$ $\forall i, 1 \leq i \leq n$.

Let $s(x) = \begin{cases} m(f \mid_{[x_{i-1},x_i]} & \text{if } x \in [x_{i-1}, x_i)) \\ f(b) \text{ if } x = b \end{cases}$

Let $f(x) = \begin{cases} M(f \mid_{[x_{i-1},x_i]} & \text{if } x \in [x_{i-1}, x_i)) \\ f(b) \text{ if } x = b \end{cases}$

Then $s \leq f \leq t$ and

$$\begin{aligned}
\int_a^b (t - s) &= \sum_{i=1}^n \left(M\left(f \mid_{[x_{i-1},x_i]}\right) - m\left(f \mid_{[x_{i-1},x_i]}\right)\right) \\
&= \sum_{i=1}^n span\left(f \mid_{[x_{i-1},x_i]}\right)(x_i - x_{i-1}) \\
&< \sum_{i=1}^n \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\
&< \frac{\epsilon}{b-a} \sum_{i=1}^n (x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a}(b - a) \\
&= \epsilon
\end{aligned}$$

Concluded by C.5 on HW5 □

Theorem 13.2. *If $f : [a, b] \to \mathbb{R}$ integrable then $g(x) = \int_a^x f$ $(g : [a, b] \to \mathbb{R})$ is continuous.*

Proof. By def of integrable, f bdd. Say $|f(x)| \leq K\ \forall x \in [a, b]$. To show g continuous given $\epsilon > 0$, take $\delta = \frac{\epsilon}{K}$. Then $|y - x| < \delta$

$$\begin{aligned}\Rightarrow |g(y) - g(x)| &= \left|\int_a^y f - \int_a^x f\right| \\ &= \left|\int_x^y f\right| \\ &\leq \int_x^y |f| \\ &\leq K \\ &\leq \int_x^y K = (y - x)K < \delta K = \frac{\epsilon}{K}K = \epsilon\end{aligned}$$

□

13.2 Bolzano + IVT

Theorem 13.3 (Bolzano)**.** *Let $f : [a, b] \to \mathbb{R}$ continuous and $f(a) < 0 < f(b)$. Then $\exists x \in [a, b]$ such that $f(x) = 0$.*

Remark. *This is false is we have a single discontinuity, or if $\mathbb{R}$ is replaced by $\mathbb{Q}$.*

Lemma 13.4. *If $f : [a, b] \to \mathbb{R}$ continuous and $f(x) > 0$, then $\exists \delta > 0$ such that $|y - x| < \delta \Rightarrow |f(y)| > 0$.*

Pf of Lemma. Take $\epsilon = f(x) > 0$. Continuity at $x \Rightarrow \exists \delta > 0 \mid |y - x| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

$$\begin{array}{rcccl} -f(x) & < & f(x) - f(y) & < & f(x) \\ -f(x) & < & -f(x) + f(y) & < & f(x) \\ 0 & < & f(y) & < & 2f(x) \end{array}$$

□

Pf of Thm. Let $S = \{x \in [a, b] \mid f(x) < 0\}$ at $S \Rightarrow S$ nonempty. S bdd above by $b \Rightarrow \sup S$ exists.

Assume $f(c) > 0$

Then $c > a$. Apply lemma to f at $c \Rightarrow \exists \delta > 0 \mid |y - c| < \delta$. In particular $\exists y < c$ with $f(y) > 0$. But then y is an upper bd for S; contradiction.

Assume $f(c) < 0$

Then $c < b$. Lemma applied to $-f$ at $c \Rightarrow \exists \delta > 0 \mid |y - c| < \delta \Rightarrow f(y) < 0$. In particular $\exists$ some $y > c$ with $f(y) < 0$, so $y \in S$, contradicting $c = \sup S$. Therefore $f(c) = 0$. □

Theorem 13.5 (Intermediate Value Theorem)**.** *Suppose* $g : [a,b] \to \mathbb{R}$ *continuous.* $g(a) < K < g(b)$. *Then* $\exists c \in [a,b] \mid g(c) = K$.

Proof. Apply Bolzano to $f(x) = g(x) - K$. □

13.3 Inverses

Proposition. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous and* *strictly* *increasing. Then* f *induces a bijection.* $f : [a,b] \to [c,d]$ *where* $c = f(a)$ *and* $d = f(b)$ *and the inverse function* $f^{-1} : [c,d] \to [a,b]$ *is also continuous and strictly increasing.*

"Pf". Inj. by strictly incr.
Surj. by IVT
Continuity: See apostol □

Ex $\quad f(x) = x^n$
$f : [0,b] \to \mathbb{R} \Rightarrow$ inverse function $f^{-1}\left(x^{\frac{1}{n}}\right) = \sqrt[n]{x}$

14 October 21, 2019

14.1 Derivatives

Given $f : [a,b] \to \mathbb{R}$ and $x \in [a,b]$, we say f is differentiable at x if $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. If so, the derivative $f'(x)$ is the value. Also note that we can write this as $\lim_{y\to x} \frac{f(y)-f(x)}{y-x}$.
Ex
$f(x) = c$

$$\lim_{h\to 0} \frac{c-c}{h} = 0$$

$f(x) = \alpha x$

$$\lim_{h\to 0} \frac{\alpha(x+h) - \alpha(x)}{h}$$

$$\lim_{h\to 0} \frac{\alpha h}{h} = \alpha$$

$f(x) = x^2$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} 2x + h = 2x$$

$f(x) = \sqrt{x} \quad x > 0$

$$\lim_{y \to x} \frac{\sqrt{y} - \sqrt{x}}{y - x} = \lim_{y \to x} \frac{\sqrt{y} - \sqrt{x}}{(\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})}$$

Counter-Ex

$f(x) = |x|$ at $x = 0$ and $f(x) = \begin{cases} \sqrt{x} \;\; x \geq 0 \\ -\sqrt{-x} \;\; x \leq 0 \end{cases}$.

Ex

$f(x) = \begin{cases} 0 \;\; x \leq 0 \\ x^2 \;\; x \geq 0 \end{cases}$ is still differentiable.

Theorem 14.1. *f diff at $x \Longrightarrow$ continuous at x*

Proof.

$$\begin{aligned} \lim_{h \to 0} (f(x+h) - f(x)) &= \left(\lim_{h \to 0} h \right) \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) \\ &= 0 \cdot f'(x) \\ &= 0 \end{aligned}$$

□

Theorem 14.2. *Suppose $f : [a,b] \to \mathbb{R}, g : [a,b] \to \mathbb{R}$ diff at $x \in [a,b]$. Then so are $f+g, f-g, f \cdot g, f/g$ if $g(x) \neq 0$ and*

1. $(f+g)' = f' + g'$
2. $(f-g)' = f' - g'$
3. $(fg)' = fg' + f'g$
4. $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

Proof. .

1. $$\lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

   $$(f+g)' = f'(x) + g'(x)$$

2. Similar

3.

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h} \right) \\
&= \lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= g(x) \cdot f'(x) + f(x) \cdot g'(x)
\end{aligned}$$

4. First compute $\left(\frac{1}{g}\right)'$

$$\begin{aligned}
\left(\frac{1}{g}\right)'(x) &= \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{g(x) - g(x+h)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x) - g(x+h)}{h} \cdot \lim_{h\to 0} \frac{1}{g(x)g(x+h)} \\
&= -g' \cdot \frac{1}{g^2} \\
&= \frac{-g'}{g^2}(x)
\end{aligned}$$

□

Remark. *Can assume $g(x+h) \neq 0$ b/c $\exists n$ of x with $g(y) \neq 0, |x-y| < \delta$.*

15 November 18, 2019

15.1 Sequences of Functions

Let $I \subset \mathbb{R}$ be an interval. A sequence of functions is a function $f : I \times \mathbb{Z}_{\geq 0} \to \mathbb{R}$. Denote $f(x, n)$ by $f_n(x)$. Think of "list of functions" $f_0, f_1, f_2, \ldots : I \to \mathbb{R}$. We may start at different indices.

Definition. *A sequence of functions $\{f_n\}$ converges pointwise if $\forall x \in I, \lim_{n\to\infty} f_n(x) = f(x)$.*
-or-
$\forall x \in I, \forall \epsilon > 0, \exists N$ such that if $n \geq N, |f_n(x) - f(x)| < \epsilon$.

Ex
$I = [0, 1], f_n(x) = x^n$

Claim: f_n converge pointwise to $f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$

Check: if $x = 1, f_n(1) = 1^n = 1$
$\Rightarrow f_n(1) \to f(1)$

$\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$. Divergence test $\Rightarrow \lim_{n\to\infty} x^n = 0$ if $|x| < 1$
$\Rightarrow f_n(x) \to 0 = f(x)$ if $x < 1$

Definition. *A sequence of functions $\{f_n\}$ converges uniformly to f if $\forall \epsilon > 0, \exists N$ such that $\forall n \geq N, \forall x \in I, |f_n(x) - f(x)| < \epsilon$.*

Theorem 15.1. *If $\{f_n\}$ is a sequence of continuous and $f_n \to f$ uniformly, then f is also continuous.*

Proof. Need to show that if $y \in I$, then $\forall \epsilon > 0, \exists \delta > 0 \mid$ if $|x - y| < \delta$ and $x \in I$, then $|f(x) - f(y)| < \epsilon$. Given $\epsilon > 0$,

- $\exists N \mid \forall n \geq N, \forall x \in I, |f_n(x) - f(x)| < \frac{\epsilon}{3}$

Since f_N is continuous, we know that

- $\exists \delta > 0 \mid$ if $|x - y| < \delta$, then $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$

Therefore, for this δ, we know $|x - y| < \delta \Rightarrow$

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

□

Remark. *Uniform convergence $\Longrightarrow$ pointwise convergence*

15.2 Series of Functions

Definition. *A series of functions $\{f_n\}$ is the sequence of partial sums $\sum_{i=0}^{n} f_i$.*

Corollary 15.1.1. *If $\{f_n\}$ are continuous and $\sum_{n=0}^{\infty} f_n$ converges to f, then f is continuous.*

Proof. By linearity of continuous, each partial sum is continuous. Apply previous result. □

Theorem 15.2 (Integration of series)**.** *If $f_n : [a, b] \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly, and if $g_n(x) = \int_a^x f_n(t)dt$ and $g(x) = \int_a^x f(t)dt$, then $g_n \to g$ uniformly. "you can exchange $\int$ and uniform limits"*

Proof. Given $\forall \epsilon > 0$, $\exists N \mid \forall n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)}$. Then if $n \geq N$,

$$\begin{aligned}
|g_n(x) - g(x)| &= \left| \int_a^x f_n - \int_a^x f \right| \\
&= \left| \int_a^x (f_n - f) \right| \\
&\leq \int_a^x |f_n - f| \\
&\leq \int_a^x \frac{\epsilon}{2(b-a)} \\
&= \frac{\epsilon}{2(b-a)}(x-a) \\
&< \epsilon
\end{aligned}$$

□

Corollary 15.2.1. *If $\sum_{n=0}^{\infty} f_n \to f$ uniformly each $f_n : [a,b] \to \mathbb{R}$ continuous, then $\int_a^x f = \sum_{n=0}^{\infty} \int_a^x f_n$. "you can integrate term by term"*

Proof. Substitute $h_m = \sum_{n=0}^{\infty} f_n$ in above thm and note that

$$\begin{aligned}
\int_a^x h_m &= \int_a^x \sum_{n=0}^{m} f_n \\
&= \sum_{n=0}^{m} \int_a^x f_n
\end{aligned}$$

so

$$\int_a^x f = \lim_{m\to\infty} \int_a^x h_m = \lim_{m\to\infty} \sum_{n=0}^{m} \int_a^x f_n = \sum_{n=0}^{\infty} \int_a^x f_n$$

□

15.3 Weierstrass M-test

Theorem 15.3 (Weierstrass M-test). *If $f_n : I \to \mathbb{R}$ and $\forall n, \exists M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ and $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{n=0}^{\infty} f_n$ converges uniformly (and absolutely) to a limit.*

<u>Ex</u>

$$f_n(x) = \frac{\sin(nx)}{2^n}$$

Since sin is bounded, $|f_n(x)| \leq \frac{1}{2^n} = M_n$

$\Rightarrow \sum_{n=0}^{\infty} \frac{\sin(nx)}{2^n}$ is continuous (!).

Proof. Since $\sum_{n=0}^{\infty} M_n - M$ converges, $\forall \epsilon > 0, \exists N \mid \forall n \geq N$,

$$\begin{aligned}
\epsilon &> \left| M - \sum_{i=1}^{n} M_i \right| \\
&= \left| \sum_{i=n+1}^{\infty} M_i \right| \\
&\geq \sum_{i=n+1}^{\infty} |f_i(x)| \text{ by comparison test} \\
&\geq \left| \sum_{i=n+1}^{\infty} f_i(x) \right| \text{ by result on absolute convergence} \\
&= \left| f(x) - \sum_{i=0}^{n} f_i(x) \right| \text{ where } f(x) = \sum_{i=0}^{\infty} f_i(x) \\
&\Rightarrow \sum_{i=0}^{n} f_i \to f \text{ uniformly}
\end{aligned}$$

□

15.4 Power Series

Definition. *If $\{a_n\}$ is a sequence of reals, then the series $\sum_{n=0}^{\infty} a_n x^n$ is called the power series corresponding to $\{a_n\}$ centered at 0. $\sum_{n=0}^{\infty} a_n (x-c)^n$ is the same centered at c.*

16 November 20, 2019

16.1 Power Series

Definition. *The power series w/ coefficients $\{a_n\}$ centered at c is the series $\sum_{n=0}^{\infty} a_n (x-c)^n$.*

Theorem 16.1. *For any power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, exactly one of the following occurs:*

a) t converges absolutely everywhere

b) It converges only at $x = c$ (and it converges absolutely at $x = c$)

c) $\exists R > 0$ such that the series converges absolutely on $(c - R, c + R)$ and diverges if $x < c - R$ or $x > c + R$

Remark. *This R is called the radius of convergence. In 16.1.a say $R = \infty$ and in 16.1.b say $R = 0$.*

Lemma 16.2. *Assume a power series centered at c converges at x. Then it converges absolutely* $\forall y \in \mathbb{R}$ *such tht* $|y-c| < |x-c|$.

Pf of Lemma. Assume $x \neq c, \sum_{n=0}^{\infty} a_n(x-c)^n$ converges. Divergence Test $\Rightarrow$ $\lim a_n(x-c)^n = 0$. $\exists N \mid \forall n \geq N, |a_n(x-c)^n| < 1$. Set $z = \frac{|y-c|}{|x-c|} < 1$ and $z \geq 0$. If $n \geq N, |a_n(y-c)^n| = |a_n(x-c)^n| z^n < z^n$. This implies that $\sum_{n=0}^{\infty} |a_n(y-c)^n|$ converges by comparison to $\sum_{n=0}^{\infty} z^n$. □

Pf of 16.1. Clearly, a), b), c) are mutually exclusive. Assume a and b don't happen. Then c does. Let $S = \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} s_n(x-c)^n$ absolutely converges$\}$. Note:

- $c \in S$
- If $x \in S$ and $R = |x-c|$, then lemma $\Rightarrow x \in (c-R, c+R) \subseteq S$
- If $x \notin S$ and $R = |x-c|$, then $S \subseteq [c-R, c+R]$ by contrapositive of lemma.

S is nonempty, and by "not a" and above note, S is bounded $\Rightarrow \sup S$ exists. Let $R := \sup S - c$.

$$\begin{aligned} \text{“not b”} &\Rightarrow \exists x \in S \mid x \neq c \\ &\Rightarrow \sup S > c \\ &\Rightarrow R > 0 \end{aligned}$$

If $|x-c| > R$, then $\exists y$ with $y - c \in (R, |x-c|) \Rightarrow y > c + R = \sup S \Rightarrow y \notin S$. And so by notes, $x \notin S$. Hence $S \subseteq [c-R, c+R]$. On the other hand, consider a point inside the interval such that $|x-c| < R$, then $c + |x-c| < c + R = \sup S$. Approximation property $\Rightarrow \exists y \in S$ with $c + |x-c| < y < c + R$. Thus, $|x-c| < y - c$. Lemma $\Rightarrow x \in S \Rightarrow (c-R, c+R) \subseteq S$. □

Definition. $0! = 1$. *If* $n \in \mathbb{Z}_{\geq 0}, n! = (n-1)! \cdot n$ *(Inductive def).*

16.2 Power Series of sin and cos

Definition. *Define the following two power series:*

$$\sin(x) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

$$\cos(x) := \sum_{n=1}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!}$$

Then,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Note, $\sin(0) = 0, \cos(0) = 1, \sin(-x) = -\sin x, \cos(-x) = \cos(x)$.

16.3 Convergence

Proposition. *Let* $\sum_{n=0}^{\infty}(a_n)(x-c)^n$ *be a power series with* $R > 0$ *and let* $[a,b] \subseteq (c-R, c+R)$*. Then the partial sums converge uniformly on* $[a,b]$*.*

Proof. $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges if $x \in [a,b]$. Divergence test $\Rightarrow \lim_{n\to\infty} a_n(x-c)^n = 0$ if $x \in [a,b]$. $\exists y$ with $y \in (c-R, c+R)$. Then, $|y-c| > |a-c|$ and $|y-c| > |b-c|$. Let

$$D = \max\{|b-c|, |a-c|\} \qquad z = \frac{\max\{|b-c|, |a-c|\}}{|y-c|}$$

We know that $z < 1$ and $z \geq 0$. So, $z = \frac{D}{|y-c|}$. $\exists N \mid \forall n \geq N, |a_n(y-c)^N| < 1$. We know this because $a_n(y-c)^n$ tends to 0.

$$|a_n(x-c)^n| \leq |a_n \cdot D^n| \leq |a_n(y-c)^n| \cdot z^n \leq z^n \; \forall n \geq N$$

Let $M_n = z^n$. Then M-test $\Rightarrow$ uniform convergence. □

Corollary 16.2.1. $\sum_{n=0}^{\infty} a_n(x-c)^n$ *is continuous on the Interval of Convergence.*

Proof. Use continuity of polynomials and the above Prop. □

Remark. *16.2.1 implies* $\sin x, \cos x$ *are continuous.*

Theorem 16.3. *Assume that* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *(*f *is given by a power series) on* $(c-R, c+R)$*.*

1. $\int_c^x f = \sum_{n=0}^{\infty} \int_c^x a_n(t-c)^n dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$
2. f *is differentiable in* $(c-R, c+R)$ *and* $f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n(x-c)^n) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$

Proof. Do each separately.

1. f is continuous $\Rightarrow$ f satisifes uniform convergence $\Rightarrow$ prior results □
2. Let $c = 0$ for simplicity. Consider $\sum_{n=1}^{\infty} a_n n(x)^{n-1}$. We want to show that this converges in $(-R, R)$. Choose x with $x \in (-R, R)$ and let h be such that $x + h \in (-R, R)$. $f(x), f(x+h)$ are given by convergent series so
$$\frac{f(h) - f(x+h)}{h} = \sum_{n=0}^{\infty} a_n \frac{(x+h)^n - x^n}{h}$$
MVT applied to x^n on $(x, x+h) \Rightarrow$ there is a point c_n with $c_n \in (x, x+h)$.
$$\frac{(x+h)^n - x^n}{h} = nc_n^{n-1}$$
$$\frac{f(x+h) - f(x)}{h} = \sum a_n x_n^{n-1} \geq \sum na_n x^{n-1}$$
Thus, $\sum na_n x^{n-1}$ converges by comparison.

□

17 November 25, 2019

17.1 Theorem from last class

Theorem 17.1. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *in* $(-R, R)$. *Then*

1. $\sum_{n=1}^{\infty} n a_n x^{n-1}$ *converges in* $(-R, R)$
2. $f'(x)$ *exists and equals* $\sum_{n=1}^{\infty} n a_n x^{n-1}$ *in* $(-R, R)$

Proof of a. First assume $0 < x < R$. Pick an h such that $x < x + h < R$.

$$\frac{f(x+h) - f(x)}{h} = \sum a_n \frac{(x+h)^n - x^n}{h}$$

MVT applied to x^n in $(x, x+h) \Rightarrow \exists c_n$ such that $x < c_n < x+h$ and $\frac{(x+h)^n - x^n}{h} = n a_n c_n^{n-1}$.

$$\sum_{n=1}^{\infty} a_n n c_n^{n-1} = \frac{f(x+h) - f(x)}{h}$$

$\sum_{n=1}^{\infty} a_n n c_n^{n-1}$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n n x^{n-1}$ converges also since $x < c_n$.

Similar if $x < 0$ □

Proof of b. Let $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Thm about term by term integration

$$\int_0^x g(t) dt = \sum_{n=1}^{\infty} a_n x^n + C$$

g continuous $\Rightarrow f'(x)$ exists and equals $g(x)$. □

Corollary 17.1.1. *Convergent power series can be differentiated as many times as you like (<u>smooth</u>).*

<u>Ex</u>

$$\sin'(x) = \cos(x)$$
$$\cos'(x) = \sin(x)$$

This can be found by differentiating term by term.

<u>Ex</u>

$$(\cos x)^2 + (\sin x)^2 = 1$$

Proof. Let $h(x) = (\cos x)^2 + (\sin x)^2$.

$$\begin{aligned} h'(x) &= 2 \cos x (\cos' x) + 2 \sin x (\sin' x) \\ &= -2 \cos x \sin x + 2 \sin x \cos x \\ &= 0 \end{aligned}$$

$h(0) = 1^2 + 0^2 = 1$. Thus, $h(x) = 1 \; \forall x$. □

17.2 Limits of Power Series

Ex Prop.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Proof. $\frac{\sin x}{x}$ is given by a power series unless $x = 0$.

$$\begin{aligned} \frac{\sin x}{x} &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \dots\right) \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \end{aligned}$$

This is a continuous function on $(-\infty, \infty)$. So,

$$\lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \dots\right) = 1 - \frac{0^2}{3!} + \dots = 1$$

Therefore $\lim_{x\to 0} \frac{\sin x}{x} = 1$ since they are equal when $x \neq 0$. □

Remark. *In general, one can "compute limits inside power series" at $x = c$ with this method.*

17.3 Differentiation of Power Series

Say we have a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \implies f(c) = a_0$$

$$f'(x) = \sum_{n=1}^{\infty} a_n n (x-c)^{n-1} \implies f'(c) = a_1$$

$$f''(x) = \sum_{n=2}^{\infty} a_n n(n-1)(x-c)^{n-2} \implies f''(c) = 2a_n$$

$$\vdots$$

$$f^{(k)}(c) = k! a_k$$

$$a_k = \frac{f^{(k)}(c)}{k!}$$

Therefore,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Corollary 17.1.2. *Uniqueness of power series:*
If $f(x) = \sum a_n(x-c)^n = \sum b_n(x-c)^n$, then $a_n = b_n$.

Proof. They are both $\frac{f^{(n)}(c)}{n!}$. □

17.4 Taylor Series

Definition. *A Taylor Series is a power series you get from some f via $a_n = \frac{f^{(n)}(c)}{n!}$.*

Ex What is the Taylor Series of e^x at $c = 0$?

$$a_n = \frac{\exp^{(n)}(0)}{n!}$$

$$\begin{aligned} &e^0 = 1 \\ &\exp' x|_{x=0} = \exp x|x = 0 = 1 \\ &\exp'' x|x = 0 = \exp x|_{x=0} = 1 \end{aligned}$$

So, Taylor Series of $\exp x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Definition. *A function on some interval I is analytic if $\forall c \in I, \exists R > 0$ such that f is represented by a power series in $(c-R, \overline{c+R})$. If it is, the power series is the Taylor series.*

Proposition. *Sums, differences, products, quotients if dem $\neq 0$, compositions of smooth/analytic functions are smooth/analytic respectively.*

Proof. Only difficult part is analytic. We'll skip. □

Definition. *If $f : I \to \mathbb{R}$ is smooth at c, its Taylor Series at c is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)$$

17.5 Problems for Smooth Functions

1. Taylor Series may not converge for $x \neq c$
2. Even if Taylor Series converges, it may not converge back to $f(x)$

Ex

$f(x) = \begin{cases} e^{\frac{-1}{x^2}} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ HW $\Rightarrow f(x)$ is smooth and $f^{(n)}(0) = 0 \, \forall n$

Taylor Series is $\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$ so $f(x)$ is not analytic.

$f^{(0)} = f$ by definition

Is $\exp x$ analytic? To solve this, use

Theorem 17.2 (Taylor's Theorem)**.** *Let f be smooth on $(c-R, c+R)$. Then, $\forall N \geq 0$, we have*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n + E_N(x)$$

where

$$E_N(x) = \frac{1}{N!}\int_c^x (x-t)^N f^{(N+1)}(t)dt$$

So Taylor Series of f converges at $x \iff E_N(x) \to 0$ as $N \to \infty$.

Proof. By Induction on N

$N = 0$ is the statement

$$f(x) = f(c) + \int_c^x f'(t)dt$$

i.e. the FTC

Assume known for N. Then,

$$E_{N+1}(x) = E_N(x) - \frac{f^{(N+1)}(c)}{(N+1)!}(x-c)^{N+1}$$

By Induction,

$$\begin{aligned} E_{N+1}(x) &= \frac{1}{N!}\int_c^x (x-t)^N f^{(N+1)}(t)dt - \frac{f^{(N+1)}(c)}{N!}\int_c^x (x-t)^N dt \\ &= \frac{1}{N!}\int_c^x (x-t)^N \left(f^{(N+1)}(t) - f^{(N+1)}(c)\right) dt \end{aligned}$$

Let $u = f^{(N+1)}(t) - f^{(N+1)}(c)$ and $dv = (x-t)^N$. By IBP

$$\begin{aligned} E_{N+1}(x) &= u(t)v(t)|_{t=c}^{t=x} - \frac{1}{N!}\int_c^x v du \\ &= 0 - \frac{1}{N!}\int_c^x \frac{-(x-t)^{N+1}}{N+1} f^{(N+2)}(t)dt \\ &= \frac{1}{(N+1)!}\int_c^x (x-t)^{N+1} f^{(N+2)}(t)dt \end{aligned}$$

□

18 December 2, 2019

18.1 Radius of Convergence

Question: is $\exp(x)$ analytic?

Recall: f <u>analytic</u> on I means $\forall c \in T, \exists R > 0$ such that the Taylor Series at c gives back f on $(c-R, c+R)$.

Proposition. *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ have radius of convergence R. If $y \in (c-R, c+R)$, then the Taylor Series of f with center y has radius of convergence at least* $\min(|y-c+R|, |y-c-R|)$.

Corollary 18.0.1. *f is analytic on $(c-R, c+R)$.*

Note: We know

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^k$$

Proof.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n(x-c)^n && (1)\\ &= \sum_{n=0}^{\infty} a_n(x-y+y-c)^n && (2)\\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}(x-y)^{n-k}(y-c)^k && (3)\\ &= \sum_{k=0}^{\infty} \left(\left(\sum_{n=k}^{\infty} a_n \binom{n}{k}(x-y)^{n-k} \right)(y-c)^k \right) && (4) \end{aligned}$$

For $(1)-(3), f(x)$ converges absolutely when $-R < x-c < R$. Thus, when absolute value signs are added, the function still converges.

$$\begin{aligned} \sum_{n=0}^{\infty} \left| a_n \sum_{k=0}^{n} \binom{n}{k}(x-y)^{n-k}(y-c)^k \right| &\le \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |x-y|^{n-k}|y-c|^k \\ &= \sum_{n=0}^{\infty} |a_n||z-c|^n \text{ where } z = |x-y| + |y-c| + c \end{aligned}$$

$\sum_{n=0}^{\infty} |a_n||z-c|^n$ converges when $c-R < z < c+R$.
$\Rightarrow$ (3) converges absolutely when $c-R < z < c+R$ so we can rearrange this sum in that interval. The inequality holds if $-R < y-c < R$ and x is within $\min(|y-c+R|, |y-c-R|)$ of y. □

18.2 Exp Analytic

<u>Ex</u> $\exp(x)$ is analytic

$$\text{T.S. at } 0 = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

By Taylor's theorem, to show this converges back to $\exp(x)$, we need $E_n(x) \to 0$.

$$E_N(x) = \frac{1}{N!}\int_0^x (x-t)^N e^t dt$$

$(x-t)^N \le x^N$ and $e^t \le e^x$. Thus,

$$E_N(x) \le \frac{1}{N!}x^N e^x x \to 0 \text{ as } N \to \infty (\text{fixed } x)$$

So Taylor Series at $0 = \exp(x)\ \forall x \in \mathbb{R}$. Proposition $\Rightarrow$ exp is analytic!

18.3 Other Examples

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ when } |x| < 1$$

Plug in x^2 for x

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} \text{ when } |x| < 1$$

Plug in $-x$ for x

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x} \text{ when } |x| < 1$$

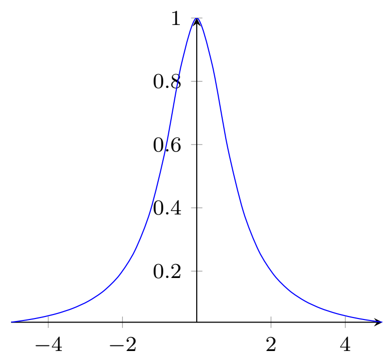
Integrate both sides:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \log(1+x) \text{ when } |x| < 1$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

<u>Ex</u>

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad |x| < 1$$

We know this is smooth on $\mathbb{R}$!



18.4 Fun Stuff

Theorem 18.1 (fun). $e \notin \mathbb{Q}$

Proof. We know $\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. It suffices ot show $\frac{1}{e} \notin \mathbb{Q}$. Let S_n be the n$^{\text{th}}$ partial sum. Terms are alternating and decreasing in abs. value.

At each step, n$^{\text{th}}$ error < n$^{\text{th}}$ term

$$0 < \left| \frac{1}{e} - s_{n-1} \right| < \frac{1}{n!}$$

Odd Terms:

$$0 < \frac{1}{e} - s_{2n-1} < \frac{1}{(2n)!}$$

$$0 < (2n-1)! \left(\frac{1}{e} - s_{2n-1} \right) < \frac{1}{2n} \leq 1n$$

Suppose $\frac{1}{e} = \frac{p}{q}$ $p, q \in \mathbb{R}$. Pick $n \geq \frac{q+1}{2}$. So, $(2n-1)!\frac{1}{e}$ is an integer. Also, $(2n-1)s_{2n-1}$ is an integer by the power series formula.

$\Rightarrow (2n-1)! \left(\frac{1}{e} - s_{2n-1} \right)$ is an integer strictly between 0 and $\frac{1}{2}$.
Contradiction! □

19 December 4, 2019

19.1 Final Exam

- Review Sessions (Sometime)
- Monday the 16th

19.2 Linear Algebra

Given functions f, g and constants c, d, you can say

$$(cf + dg)' = cf' + dg' \quad \text{(Linearity)}$$

$$\int (cf + dg) = c \int f + d \int g$$

$$\lim_{x \to c} (cf(x) + dg(x)) = c \lim_{x \to c} f(x) + d \lim_{x \to c} g(x)$$

$$\lim_{n \to \infty} (ca_n + db_n) = c \lim_{n \to \infty} a_n + d \lim_{n \to \infty} b_n$$

19.3 Fields

Definition. *A <u>field</u> is a set F together with two binary operations $+, \bullet$ and two elements $0, 1$ with $0 \neq 1$ such that:*

1. $\forall x, y \in F,\ x + y = y + x$ *and* $x \cdot y = y \cdot x$ *(Commutativity)*
2. $\forall x, y, z,\ x + (y + z) = (x + y) + z$ *and* $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *(Associativity)*
3. $\forall x, y, z,\ x \cdot (y + z) = x \cdot y + x \cdot z$ *(Distributivity)*
4. $\forall x,\ x + 0 = x$ *and* $x \cdot 1 = x$
5. $\forall x, \exists y$ *such that* $x + y = 0$. *We call this y "$-x$".*
6. $\forall x \neq 0, \exists y$ *such that* $x \cdot y = 1$ *(call this y "$\frac{1}{x}$" or "x^{-1}")*

<u>Ex</u> $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (complex numbers - soon)
$\mathbb{F}_p = \{0, 1, 2, \ldots, p^{-1}\}$ with "modular arithmetic". $p =$ a prime number.

<u>Ex</u> A thing that seems bad
Let F be a field. Define the following.

$$\begin{aligned} 2 &= 1 + 1 \\ 3 &= 2 + 1 \\ 4 &= 3 + 1 \\ &\vdots \end{aligned}$$

Then it may happen that $p = 0$ for p a prime. Eg. $\mathbb{F}_p$. So we can't always divide by n, n a natural number.

19.4 Vector Spaces

Definition. *Let F be a field. A* <u>*vector space*</u> *over F is a set V together with two operations:*

$+ : V \times V \to V$ (*Addition*)

$\bullet : F \times V \to V$ (*Scalar Multiplication*)

and an element $0 \in V$, such that

1. $\forall x, y \;\; x + y = y + x$
2. $\forall x, y, z \;\; x + (y + z) = (x + y) + z$
   $\forall a, b \in F \;\; a \cdot (b \cdot x) = (a \cdot b) \cdot x$
3. $c \in F, x, y \in V \;\; c \cdot (x + y) = c \cdot x + c \cdot y$
   $c, d \in F, x \in V \;\; (c + d) \cdot x = c \cdot x + d \cdot x$
4. $\forall x,\ x + 0 = x$ *and* $1 \cdot x = x$
5. $\forall x, \exists y \mid x + y = 0$

Remark. *Call elements of V "vectors" and call elements of F "scalars". For some reason, Apostol calls vector spaces "linear spaces".*

Proposition (Easy facts about vector spaces)**.** *Let F be a field, V a vector space over F.*

1. $\forall x \in V,\ (-1) \cdot x$ *is the unique y such that $x + y = 0$. (call it "$-x$")*
2. $\forall x \in V,\ 0_F \cdot x = 0_V$.
   $\forall c \in F,\ c \cdot 0_V = 0_V$.
3. $\forall x \in F, \forall x \in V,\ (-c) \cdot x = c \cdot (-x) = -c \cdot x$.
4. *If $c \cdot x$, then $c = 0$ or $x = 0$.*
   If $c \cdot x = c \cdot y$, then $c = 0$ or $x = y$.
   If $c \cdot x = d \cdot x$, then $c = d$ or $x = 0$.

Proof of 2. Assuming 1,

$$\begin{aligned} 1 + (-1) &= 0 \\ (1 + (-1)) \cdot x &= 0 \cdot x \\ 1 \cdot x + (-1) \cdot x &= 0 \cdot x && \text{Distributivity} \\ x + (-1) \cdot x &= 0 \cdot x && \text{Identity of 1} \\ 0 &= 0 \cdot x \end{aligned}$$

□

19.5 Examples

Given F a field, let

$$\begin{aligned} F^n &= \{cx_1, cx_2, cx_3, \dots, cx_n \mid x_1, x_2, \dots, x_n \in F\} \\ &= \{\text{functions } [n] \to F\} \text{ (formally)} \end{aligned}$$

Let $0_{F^n} = (0, 0, \dots, 0)$.

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

$$c \cdot (x_1, \dots x_n) = (c \cdot x_1, c \cdot x_2, \dots, c \cdot x_n)$$

Check: this satisfies all axioms. E.g. Associativity of $+$:

$$\begin{aligned} ((x_1, \dots, x_n) + (y_1, \dots, y_n)) + (z_1, \dots, z_n) &= (x_1 + y_1, \dots, x_n + y_n) + (z_1, \dots, z_n) \\ &= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n) \\ &= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \\ &= (x_1, \dots, x_n) + (y_1 + z_1, \dots, y_n + z_n) \\ &= (x_1, \dots, x_n) + ((y_1, \dots, y_n) + (z_1, \dots, z_n)) \checkmark \end{aligned}$$

19.6 Notation

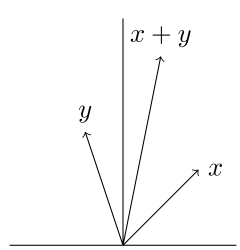
If $x = (x_1, \dots, x_n) \in F^n$, then we call $x_i \in F$ the $\underline{\text{i}^{\text{th}} \text{ component}}$ of x. So two vectors are equal $\iff$ all components are equal.

Ex Let $F^0 = \{0\}$. This is a vector space.

Ex F^1 "is" F as a set. F^1 "forgets indentity" 1.

19.7 Visualization

Take $F = \mathbb{R}$. $\mathbb{R}^2$ = a plane and $\mathbb{R}^3$ = 3 - dim'l space.



Ex Let F be a field and S a set. Denote the set of functions $S \to F$ by $\mathcal{F}(S, F)$. 0 is the zero function.

$$(f + g)(s) = f(s) + g(s)$$

$$(cf)(S) = c \cdot f(s)$$

e.g. if $F = \mathbb{R}, S = \mathbb{R}$. $\mathcal{F}(\mathbb{R}, \mathbb{R})$ = space of all real valued functions on $\mathbb{R}$. More generally, take set S and vector space V (over F). $\mathcal{F}(S, V)$ is a vector space. e.g. functions $\mathbb{R} \to \mathbb{R}^2$.

20 December 9, 2019

20.1 Final Exam Stuff

Review Sessions

1. Tomorrow 12/10 5-7 room 520
2. TBD

Final Exam
Exactly one week in usual place. Warner will also hold office hours sometime TBD on Friday. Practice problems will be posted.

20.2 Linear Alg Cont.

Functions are to sets as linear maps are to vector spaces.
Ex

- $\mathbb{R}^n$
- $\mathcal{F}(S, F)$ S set, F field
- $\mathcal{F}(S, V)$ V vector space

Ex Check distributivity axiom for $\mathcal{F}(S, V)$.

$$\begin{aligned}
(c(f+g))(s) &= c \cdot (f+g)(s) \text{ by defn of scalar multiplication} \\
&= c \cdot (f(s) + g(s)) \text{ by defn of vector addition} \\
&= cf(s) + cg(s) \text{ by distributivity in } V \\
&= (c \cdot f)(s) + (c \cdot g)(s) \\
&= (c \cdot f + c \cdot g)(s)
\end{aligned}$$

20.3 Subspaces

Definition. *Let V be a vector space/F. A nonempty subset $W \subseteq V$ is a subspace if $x, y \in W \Rightarrow x + y \in W$ and $x \in W, c \in F \Rightarrow cx \in W$.*

Observation: If $W \subseteq V$ is a subspace, then it is a vector space itself (with induced $+, \bullet$).

Proof. $+, \bullet$ well defined by the defn. All identities follow from those in V. Let $w \in W$. Then $-w \in W$ and $w + (-w) \in W \Rightarrow O_v \in W$. □

Ex

- $W = \{0\}$
- $W = V$
- Any line through the origin in $\mathbb{R}^2$
- NOT any line not through origin in $\mathbb{R}^2$
- A plane going through the origin in $\mathbb{R}^2$

Ex
V vector space, $v \in V$ with $v \neq 0$. $W = \{cv \mid c \in F\}$ is a subspace.

Ex

- $V = \mathcal{F}(\mathbb{R}, \mathbb{R}) \supset \{\text{bounded functions}\}$ and $\mathcal{F}(\mathbb{R}, \mathbb{R}) \supset \{\text{continuous functions}\}$
- $V = \mathcal{F}([a, b], \mathbb{R}) \supset \{\text{bounded functs}\} \supset \{\text{integrable functs}\} \supset \{\text{continuoud functs}\} \supset \{\text{differentiable functs}\} \supset \{\text{smooth functs}\} \supset \{\text{analytic functs}\} \supset \{\text{polynomials}\} \supset \{\text{constant functions}\}$.

Proposition. *Let V be a vector space. $W_1 \subseteq V, W_2 \subseteq V$ subspaces. Then $W_1 \cap W_2$ is a subspace.*

Proof. If $x, y \in W_1 \cap W_2$, then $x, y \in W_1$ and $x, y \in W_2$ so $x + y \in W_1$ and $x + y \in W_2$ so $x + y \in W_1 \cap W_2$. If $x \in W_1 \cap W_2$ and $c \in F$, then $x \in W_1$ and $x \in W_2$ so $cx \in W_1$ and $cx \in W_2$ so $cx \in W_1 \cap W_2$. □

Remark. *Unions of subspaces are generally not subspaces.*

Definition. *If V, W are vector spaces over F, then a linear map $V \to W$ is a function $T : V \to W$ such that $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv_1) = cT(v_1)$.*

Ex derivatve, $\int$, $\lim_{x \to c}$, $\lim_{n \to \infty}$

Ex

- $\mathrm{id_v}$: $V \to V$ is linear
- If $T : U \to V$ is linear and W is a subspace of U, then the restriction $T\mid_W : W \to V$ is linear.
- If $W \subseteq U$ is a subspace, $\mathrm{id_U}|_W : W \to U$ is linear (inclusion function).

Definition. *For $x_1, \ldots, x_n \in U, U$ v.s./F and $c_1, \ldots, c_n \in F$, the linear combination of x_i with coefficients c_i is the vector $\sum_{i=1}^n c_i x_i$.*

Proposition. *$T : U \to V$ is linear $\iff \forall x_1, \ldots x_n \in U, \forall c_1, \ldots, c_n \in F$,*

$$T\left(\sum_{i=1}^n c_i x_i\right) = \sum_{i=1}^n c_i T(x_i)$$

Remark. *"linear = preserves linear combination"*

"Proof". Induct on n. □

20.4 Kernal and Image

Definition. *Suppose $T : U \to V$ is linear. Denote $T^{-1}(\{0\})$ by* $\ker T$ *"kernal of T". Denote $T(U)$ by* $\operatorname{im} T$ *"image of T".*

Proposition. $\ker T$ *is a subspace of U.* $\operatorname{im} T$ *is a subspace of V.*

Proof. Just have to check closure under $+, \bullet$.

$$\begin{aligned} U_1, U_2 \in \ker T &\Rightarrow T(U_1) = T(U_2) = 0 \\ &\Rightarrow T(U_1 + U_2) = T(U_1) + T(U_2) \\ &\Rightarrow U_1 + U_2 \in \ker T = 0 \end{aligned}$$

$U \in \ker T, c \in F, T(u) = 0$.

$$\begin{aligned} T(cu) = cT(u) = c \cdot 0 = 0 &\Rightarrow cu \in \ker T \\ &\Rightarrow \ker T \text{ is a subspace} \end{aligned}$$

$v_1, v_2 \in \operatorname{im} T \Rightarrow \exists u_1, u_2 \mid T(u_1) = v_1$ and $T(u_2) = v_2$.

$$T(u_1 + u_2) = T(u_1) + T(u_2) = v_1 + v_2 \Rightarrow v_1 + v_2 \in \operatorname{im} T$$

$v \in \operatorname{im} T, c \in F \Rightarrow \exists u \mid T(u) = v$ and $T(cu) = cT(u) = cv \Rightarrow cv \in \operatorname{im} T$. □

Ex
$F = \mathbb{R}, V = \{\text{differentiable functions on } \mathbb{R}\}, D : V \to \mathcal{F}(\mathbb{R}, \mathbb{R}), D(f) = f'$. What is $\ker D$? Calculus $\Rightarrow \ker D = \{\text{constant functions}\}$. $\operatorname{im} D =$??. By FTC we do know that $\operatorname{im} D$ contains all continuous functions.

Proposition. $\ker T = \{0\} \Leftrightarrow T$ *is injective*

Proof. HW next year (!) *as in next semester* □

Note
$$\operatorname{im} T = V \iff T \text{ is surjective}$$

Definition. *An isomorphism $T : V \to V$ is a linear map that is bijective.*

21 Final Review: December 10, 2019

21.1 Notes from HW 12

"it was really hard".
Questions 9 - 10
The goal is to build a smooth function f such that $f - g$ near c, $f - h$ near d, whose g, h are smooth.

$$f = \Phi(x - c)g(x) + \Phi(x - d)h(x)$$

such that

$$\Phi(x) = \begin{cases} 1 \text{ near } 0 \\ 0 \text{ ``away from 0''} \end{cases}$$

$$\begin{aligned} f(c) &= \Phi(0) \cdot g(c) + \Phi(c-d)h(c) \\ &= 1 \cdot g(c) + 0 \cdot h(c) \\ &= g(c) \end{aligned}$$

<u>Another Question</u>
f_n uniformly converges to f. Want to show f_n are integrable $\Rightarrow$ f is also integrable.

Proof. From previous homework, we only need to prove $\forall \epsilon > 0, \exists$ two step functions $s(x), t(x)$ such that $s(x) \geq f(x) \geq t(x)$ and $\int (s(x) - t(x))dx < \epsilon$.

Mistake 1: $\{s_n\}$ to be a sequence of upper step functions of $\{f_n\}$.
$s_n(x) = \begin{cases} 1 \ \ x \in \left[\frac{2k-1}{n}, \frac{2k}{n}\right) \\ 0 \ \ x \in \left[\frac{2k}{n}, \frac{2k+1}{n}\right) \end{cases}$ As $n \to \infty, s$ is no longer a step function.
$\forall \epsilon > 0, \exists N, n \geq N \mid |f_n(x) - f(x)| < \epsilon$. Let step functions $s_N, t_N \mid s_N \geq f_N \geq t_N$ and $\int s_N - t_N < \epsilon$. Claim $s_N + \epsilon$ is an upper step function for f. $s_N + \epsilon \geq f_N + \epsilon > f$. $t_N - \epsilon$ is a lower step function for f. Let $s = s_N + \epsilon$ and $t = t_N - \epsilon$. Then,

$$\begin{aligned} \int (s-t)dx &< \int (s_N - t_N)dx + \int_a^b (2\epsilon)dx \\ &= \epsilon + 2(b-a)\epsilon = (2(b-a)+1)\epsilon \end{aligned}$$

The following should be the exam solution.
$f_n = f, \exists N, \forall n \geq N, \forall x |f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)+1}$. t_N, s_N to be step functions for $f_N(x)$ such that $\int_a^b (s(x) - t(x))dx < \frac{\epsilon}{2(b-a)+1}$. Claim $S_N + \frac{\epsilon}{2(b-a)+1} t_N - \frac{\epsilon}{2(b-a)+1}$. $\int (s-t)dx < \epsilon$. □

21.2 HW 9

Theorem 21.1. *$|f|$ is differentiable $\Rightarrow$ f is differentiable. Hint: You only need to consider x such that $f(x) = 0$. What if we consider $x_0 \mid f(x_0) > 0$.*

Proof. Mistake: Consider $f(x) > 0$. $|f|(x) = f(x) \Rightarrow f'(x)$ exists and equals $|f|'(x)$. In def of $f', f' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. $f(x) = |f|(x)$. Why $f(x+h) = |f|(x+h)$.
<u>Solution</u>
x_0 such that $f(x_0) > 0$. Goal: $\exists \delta, x \in (x_0 - \delta, x_0 + \delta)$. Know: $\exists \delta,\ f(x) > 0$. $|f(x) - f(x_0)| < \frac{f(x)}{2} \forall x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) - f(x_0) > -\frac{f(x_0)}{2} \Rightarrow f(x) > \frac{f(x_0)}{2} > 0$. □

Theorem 21.2. *f is continuous, $f(x_0) > 0, \exists (a,b) \mid x_0 \in (a,b)$ and $f(x) = 0 \, \forall x \in (a,b)$.*

21.3 Questions

1. $f(x), f(x)$ are analytic. $f(x) = g(x) \in (a,b) \Rightarrow f(x) = g(x)$ everywhere.

   Proof. $c = \sup\{x \mid f(x) = g(x)\}.h = f - g$ analytic. Taylor Expansion at c. Taylor expansion of h has infinite zeros near c. Claim series is constantly zero at c. $f - g = 0$ for $c + \epsilon \Rightarrow$ contradiction. □

2. $f : \mathbb{R} \to \mathbb{R}$ twice diff. $f(0) = 0, f(1) = 1, f(0) = f'(1) = 0$. $x \in [0,1], f''(x) \geq 2$.

   Answer. □